



Core Concepts in **Mathematics: Geometry**

Diego De Leo

2ND EDITION

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CORE CONCEPTS IN MATHEMATICS: **GEOMETRY**

Diego De Leo

2nd Edition



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Preface

In the early seventeenth century, the groundbreaking development of symbolic algebra transformed mathematics, clearing the path for a more intricate period of investigation. The tool enabled mathematicians to approach problems in a more flexible and powerful way, free from the constraints of specific numerical values.

Symbolic algebra's might resided in its capacity to perform calculations and manipulations using variables and symbols, independent of a numerical context. Moving away from concrete calculations, mathematicians could explore the underlying structures and patterns within mathematical problems, resulting in a deeper grasp of the subject. With symbolic algebra's introduction, mathematical ideas and unsolved issues became more accessible.

The shift towards analytical techniques in mathematics became more pronounced during the 19th century. Geometric methods were outdone by their more reliable and systematic counterparts, rooted in equations and functions. Subsequently, geometry's emphasis as a tool for mathematical concept communication waned in numerous textbooks. Unforeseen consequences of this shift affected how mathematics was taught and absorbed. The absence of geometry, incorporating visual and spatial elements, contributed to a lessened grasp of the subject. The difficulty many students faced in grasping abstract concepts depend on the absence of geometric insights in their analytical operation.

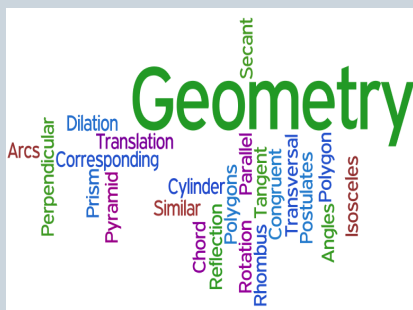
Although, our daily lives are linked to spatial, geometric, and quantitative concepts. The universe's diverse forms are highlighted by the Earth and its celestial neighbors. Each celestial body has its distinct shape and size, combining to create the elaborate universe. Spatial reasoning is a crucial element in unlocking the world's complex forms. Spatial relationships, positions, sizes, and shapes are intricately connected and require a deeper understanding within a specific context.

The importance of geometry in cultivating this spatial aptitude is fundamental. Geometry offers a system for grasping and navigating our surroundings by furnishing a basis for measurements, angle calculations, line visualization, surface comprehension, and solid structure appreciation. Geometric concepts offer us the power to calculate distances, build structures, and resolve practical issues with precision and speed. The visual and spatial aspects of geometry help cultivate creativity and intuition, serving mathematical and scientific fields well.

"Core Concepts in Geometry" 2nd edition builds a robust foundation by emphasizing critical elements. The latest edition contains revised chapters with additional topics. The limitations of students' geometric knowledge are largely attributable to deficiencies within educational institutions. Despite the availability of numerous tools for creating

and manipulating geometric shapes, many struggle to fully understand the wealth of information they contain. Comprising diverse topics, this book offers readers a launching point into central concepts of rapidly transforming geometric spheres.

OVERVIEW OF GEOMETRY



Geometry is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space. A mathematician who works in the field of geometry is called a geometer.

After studying this chapter, you will understand the core concepts of:

- Building Blocks of Geometry
- Postulates and Axiomatic Systems of Euclidean geometry

INTRODUCTION

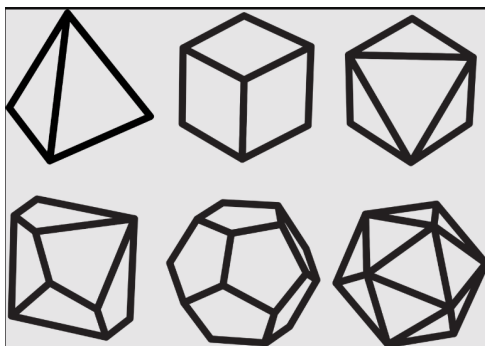
Geometry is a fundamental branch of mathematics that explores the properties, relationships, and configurations of shapes, sizes, and spaces. It plays a crucial role in understanding the physical world and has applications in various fields, from architecture and engineering to art and physics. At its core, geometry investigates the fundamental concepts of points, lines, angles, and surfaces, and how they interact to create complex structures and patterns.

One of the foundational elements in geometry is the point. A point is an exact location in space, represented by a dot. When these points are connected, they form lines, which are infinitely long and extend in both directions. Lines can intersect, creating angles where they meet. Angles are formed by two rays that share a common endpoint, known as the vertex of the angle. These angles are measured in degrees, with a full circle measuring 360 degrees.

Further building on these concepts, we encounter polygons, which are closed shapes made up of straight line segments. Triangles, for example, are polygons with three sides, while quadrilaterals have four sides. Each polygon type has its own set of characteristics, such as angles and side lengths, that define its properties. Circles are another essential element in geometry, defined as a set of points equidistant from a central point. They introduce the concept of radius, diameter, and circumference, which are integral to understanding circular shapes.

When lines and shapes come together in more complex arrangements, we delve into the realm of two-dimensional and three-dimensional geometry. Two-dimensional geometry involves flat shapes and their properties, while three-dimensional geometry incorporates depth, like solids and their volumes. The study of spatial relationships, symmetry, and transformations—such as reflections, rotations, and translations—becomes increasingly important in higher dimensions of geometry.

Geometry also provides a framework for understanding and measuring the size of objects and spaces. Concepts like area and volume allow us to quantify the amount of space enclosed by various shapes and solids. These measurements have practical applications in fields like architecture, engineering, and construction, enabling the creation of structures that are not only aesthetically pleasing but also structurally sound.



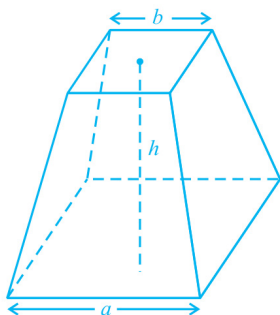


CORE CONCEPT: BUILDING BLOCKS OF GEOMETRY

Geometry appears to have originated from the need for measuring land. This branch of mathematics was studied in various forms in every ancient civilization, be it in Egypt, Babylonia, China, India, Greece, the Incas, etc. The people of these civilizations faced several practical problems which required the development of geometry in various ways.

Geometry is essentially the study of shapes. In the world around us, every object we see is a shape of some kind. Some are simple, like a triangle, square, or circle. Others seem to be combinations of these simple shapes. It is the goal of this text to guide you through the study of the simple shapes that compose nearly everything in the world, so that you can apply these concepts to more complicated problems of geometry. To begin understanding these shapes, we need to first learn their components. The simplest unit of geometry is the point. A collection of points in a certain **array** makes a line, and collections of lines in certain arrays create shapes, which may exist in a single plane, or may exist in more than one plane in space. For now it is just important to introduce you to the language of geometry. You will become familiar with the names and properties of these figures, which are the building blocks of geometry, and learn to recognize them in more complex situations.

Array is a container object that holds a fixed number of values of a single type.



For example, whenever the river Nile overflowed, it wiped out the boundaries between the adjoining fields of different land owners. After such flooding, these boundaries had to be redrawn. For this purpose, the Egyptians developed a number of

geometric techniques and rules for calculating simple areas and also for doing simple constructions. The knowledge of geometry was also used by them for computing volumes of granaries, and for constructing canals and pyramids. They also knew the correct formula to find the volume of a truncated pyramid (see Figure). You know that a pyramid is a solid figure, the base of which is a triangle, or square, or some other polygon, and its side faces are triangles converging to a point at the top. Geometry arose independently in a number of early cultures as a practical way for dealing with lengths, areas, and volumes.

Geometry began to see elements of formal mathematical science emerging in the West as early as the 6th century BC. By the 3rd Century BC, geometry was put into an axiomatic form by Euclid, whose treatment, *Euclid's Elements*, set a standard for many centuries to follow.

Geometry arose independently in India, with texts providing rules for geometric constructions appearing as early as the 3rd century BC. Islamic scientists preserved Greek ideas and expanded on them during the Middle Ages. By the early 17th Century, geometry had been put on a solid analytic footing by mathematicians such as René Descartes and Pierre de Fermat. Since then, and into modern times, geometry has expanded into non-Euclidean geometry and manifolds, describing spaces that lie beyond the normal range of human experience.

While geometry has evolved significantly throughout the years, there are some general concepts that are more or less fundamental to geometry. These include the concepts of points, lines, planes, surfaces, angles, and curves, as well as the more advanced notions of manifolds and topology or metric. Geometry has applications too many fields, including art, architecture, physics, as well as to other branches of mathematics.

Mechanism of Geometry

Geometry is one of the oldest branches of mathematics. The word geometry in the Greek language translates the words for “Earth” and “Measure”. The Egyptians were one of the first civilizations to use geometry. The Egyptians used right triangles to measure and survey land. In our modern times, geometry is used to in fields such as engineering, architecture, medicine, drafting, astronomy, and geology. We will describe two basic concepts which are a point and a line. A point is used to denote a specific location in space. Everything that we do will be viewed in two dimensions. *For example*, we could draw a point in two dimensional space and label it as point A.

No Dimensions

The most basic of all geometric terms is the point. A point is a place in space that has no length or dimension. It is represented using a dot, but this is only so that it is visible for our mathematical purposes. Although it does not have size, it does have position, and that position can be given using an ordered pair such as (x, y) .

One Dimension

If we take a step into 1-dimensional geometry, our first concept is a line, the set of all points that fall along a straight path. A line will continue along a straight path in both directions without end. Related to the line are the line segment and the ray. A **line segment** is the collection of points that lie between two specific endpoints. A ray has a starting point and then continues into infinity in one direction. These lines, segments, and rays intersect each other to form parallel and perpendicular lines and angles.

Measurement is a very common function in geometry. Since we have only one dimension at this point, you can only measure one aspect: length. Length is measured in single units such as miles, feet, inches, centimeters, etc. If we notice that a train is 300 meters long, we are noting a measure of length.

Angles have their own form of measurement. The distance between the line segments that form the angle, a curved measurement, is measured in degrees.

Two Dimensions

When we take into account two dimensions, we are introduced to the plane. A plane is a flat surface, like a piece of paper that extends forever in all directions. It has length and width but not height.

Triangles, rectangles, polygons, and circles are all common 2-dimensional shapes. With two dimensions you can measure perimeter, which is the distance around a shape.

You can also measure area. Area is the amount of space that is on the inside of a 2-dimensional shape and is given in square units. The amount of paint or carpeting it would take to cover your bedroom is a measurement that would require you to find area.

Point

A point is a way to describe a specific location in space. It is drawn by placing a single dot at the location you want to specify, and denoted in text with a capital letter.

A point in geometry is a location. It has no size i.e. no width, no length and no depth. A point is shown by a dot.

•
A

Line segment is a part of a straight line that is bounded by two distinct endpoints, and contains every point on the line that is between its endpoints.

Line

A line is determined by two distinct points and extends to infinity in both directions. Now suppose that we define two points in space and label them as A and B. We could pass a line through these points in space and the resulting line would look like the next illustration. We will label this line as $\overleftrightarrow{AB}$.

A line is defined as a line of points that extends infinitely in two directions. It has one dimension, length. Points that are on the same line are called collinear points.



A line is defined by two points and is written as shown below with an arrowhead.

$\overleftrightarrow{AB}$

Two lines that meet in a point are called intersecting lines.

A part of a line that has defined endpoints is called a line segment. A line segment as the segment between A and B above is written as:

$\overline{AB}$

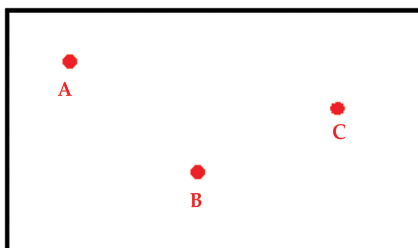
Line Segment

A line segment is part of a line that lies between two points. These two points are referred to as endpoints. In the next figure below there is an illustration of a line segment. We will label this line segment as $\overline{AB}$.

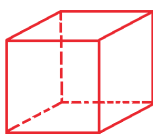


A plane extends infinitely in two dimensions. It has no thickness. An example of a plane is a [coordinate plane](#). A plane is named by three points in the plane that are not on the same line. Here below we see the plane ABC.

Coordinate plane is a tool used for graphing points, lines, and other objects. It functions like a map that follows directions from one point to another.



A space extends infinitely in all directions and is a set of all points in three dimensions. You can think of a space as the inside of a box.



Rays

A ray is a cross between a line and a line segment. It extends without bound in one direction, but not the other. It is determined by two points, one being the starting point for the ray, and the other determining the direction of the ray. A ray can be symbolized in the following way:

$$\overrightarrow{AB}$$

Below is a figure of a ray:

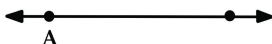


Segments and rays have no thickness, only length. They are intangible, and only used to specify a set of locations in space.



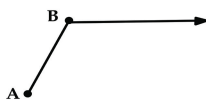
Problems

Problem: Decide whether the following is a point, line, segment, ray, or none of these.

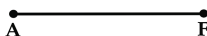


A line

Problem: Decide whether the following is a point, line, segment, ray, or none of these.



Problem: Decide whether the following is a point, line, segment, ray, or none of these.



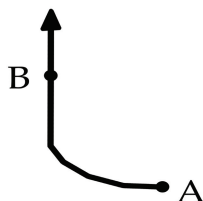
Segment

Problem: Decide whether the following is a point, line, segment, ray, or none of these.



Ray

Problem: Decide whether the following is a point, line, segment, ray, or none of these.



Curves

A curve is a 1-dimensional object that may be straight (like a line) or not; curves in 2-dimensional space are called plane curves and those in 3-dimensional space are called space curves.

Topology is concerned with the properties of space that are preserved under continuous deformations, such as stretching, crumpling and bending, but not tearing or gluing.

In **topology**, a curve is defined by a function from an interval of the real numbers to another space. In differential geometry, the same definition is used, but the defining function is required to be differentiable. Algebraic geometry studies algebraic curves, which are defined as algebraic varieties of dimension one.

Surfaces

A topology is a mathematical structure on a set that tells how elements of the set relate spatially to each other. The best-known examples of topologies come from metrics, which are ways of measuring distances between points. For instance, the Euclidean metric measures the distance between points in the Euclidean plane, while the hyperbolic metric measures the distance in the hyperbolic plane. Other important examples of metrics include the Lorentz metric of special relativity and the semi-Riemannian metrics of general relativity.

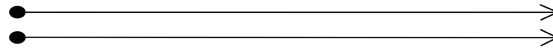
Distance

Now that we have given a basic description a line and line segment, let's use some properties of distance to find the

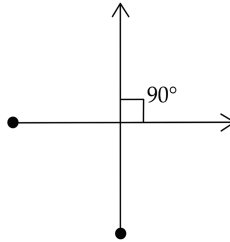
missing length of a segment. In the next example we will find the distance between two points. To find missing distances of a line segment, we use a postulate called the segment addition postulate.

Parallel and perpendicular lines

There are two types of lines that are particularly interesting and/or useful in mathematics. Parallel lines never meet or intersect. They simply go on forever side by side, a bit like railway lines:

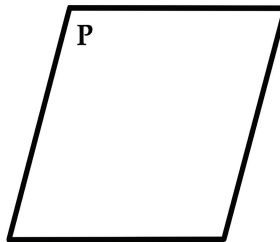


Perpendicular lines intersect at a right angle, 90° :

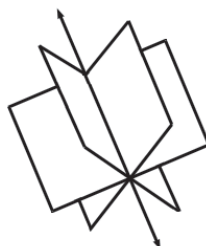


Planes

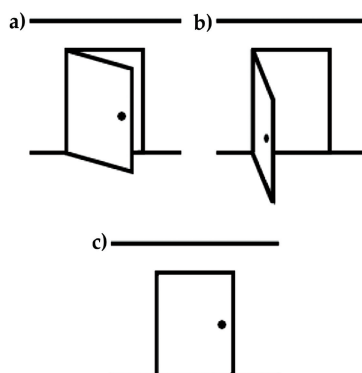
A plane is a boundless surface in space. It has length, like a line; it also has width, but not thickness. A plane is denoted by writing “plane P”, or just writing “P”. On paper, a plane looks something like this:



There are two ways to form a plane. First, a plane can be formed by three noncollinear points. Any number of collinear points form one line, but such a line can lie in an infinite number of distinct planes. See below how different planes can contain the same line.



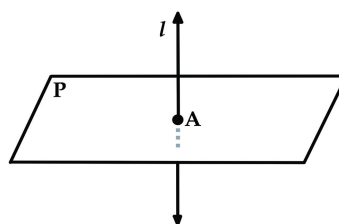
It takes a third, non-colinear point to form a specific plane. This point fixes the plane in position. The situation is something like a door being shut. Before the door is shut, it swings on hinges, which form a line. The door (a plane) can be opened to an infinite number of different positions, maybe just cracked a few inches, or maybe wide open (figures a, b in the diagram below). When the door is shut however, the wall on the other side of the hinges acts as the non-colinear third point and holds the door in place. At this point, the door represents one distinct plane (figure c).



Geometry surface

may cross itself
(and may have
other singularities),
while, in topology
and differential
geometry, it may
not.

The second way to form a plane is with a line and a point in that line. There are just two conditions. 1) The line must be perpendicular to the plane being formed (for an explanation of this concept, see [Geometric Surfaces](#), Lines and Planes); 2) the point in the line must also be in the plane being formed. Given a line, a point in that line, and these conditions, a plane is determined.



When points lie in the same plane, they are called coplanar. When points lie in different planes, they are called noncoplanar. The concept is much like that of colinearity.

Planes and Two-dimensional Shapes

A plane is a flat surface, also known as two-dimensional. It is technically unbounded, which means that it goes on forever in any given direction and as such is impossible to draw on a

page.

One of the key elements in geometry is how many dimensions you are working in at any given time. If you are working in a single plane, then it's either one (length) or two (length and width). With more than one plane, it must be three-dimensional, because height is also involved.



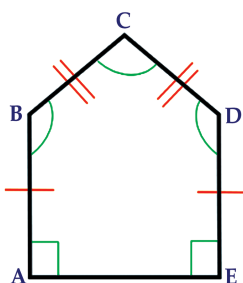
There is more about polygons in our page on Polygons. Other two-dimensional shapes include circles, and any other shape that includes a curve. You can find out more about these on our page, Curved Shapes.

Signs, Symbols and Terminology

The shape illustrated here is an irregular pentagon, a five-sided polygon with different internal angles and line lengths (see our page on Polygons for more about these shapes).

Two-dimensional shapes include polygons such as squares, rectangles and triangles, which have straight lines and a point at each corner.

- Degrees a measure of rotation - the angle between the sides.
- Angles are commonly marked in geometry using a segment of a circle (an arc), unless they are a right angle when they are 'squared off'. Angle marks are indicated in green in the example here.
- Tick marks (shown in orange) indicate sides of a shape that have equal length (sides of a shape that are congruent or that match). The single lines show that the two vertical lines are the same length while the double lines show that the two diagonal lines are the same length. The bottom, horizontal, line in this example is a different length to the other 4 lines and therefore not marked. Tick marks can also be called 'hatch marks'.
- A vertex is the point where lines meet (lines are also referred to as rays or edges). The plural of vertex is vertices. In the example there are five vertices labelled A, B, C, D and E. Naming vertices with letters is common in geometry.



The angle symbol ‘ $\angle$ ’ is used as a shorthand symbol in geometry when describing an angle. The expression $\angle ABC$ is shorthand to describe the angle between points A and C at point B. The middle letter in such expressions is always the vertex of the angle you are describing - the order of the sides is not important. $\angle ABC$ is the same as $\angle CBA$, and both describe the vertex B in this example.

If you want to write the measured angle at point B in shorthand then you would use:

$$m\angle ABC = 128^\circ \text{ (m simply means ‘measure’)}$$

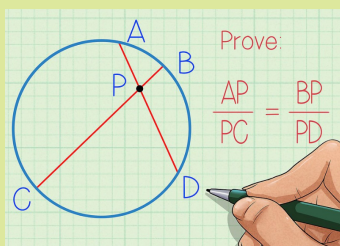
or

$$m\angle CBA = 128^\circ$$

In our example we can also say:

$$m\angle EAB = 90^\circ$$

$$m\angle BCD = 104^\circ$$



CORE CONCEPT: POSTULATES AND AXIOMATIC SYSTEMS OF EUCLIDEAN GEOMETRY

Euclidean geometry is a mathematical system attributed to the Alexandrian Greek mathematician Euclid, which he described in his textbook on geometry: the Elements.

An axiom or postulate is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments. The word comes from the Greek ‘that which is thought worthy or fit’ or ‘that which commends itself as evident.

The Greek mathematicians of Euclid’s time thought of geometry as an abstract model of the world in which they lived. The notions of point, line, plane (or surface) and so on were derived from what was seen around them. From studies of the space and solids in the space around them, an abstract geometrical notion of a solid object was developed. A solid has shape, size, position, and can be moved from one place to another. Its boundaries are called surfaces. They separate one part of the space from another, and are said to have no thickness. The boundaries of the surfaces are curves or straight lines. These lines end in points.

Consider the three steps from solids to points (solids-surfaces-lines-points). In each step we lose one extension, also called a dimension. So, a solid has three

dimensions, a surface has two, a line has one and a point has none. Euclid summarized these statements as definitions.

Book 1 of the ‘Elements’. A few of them are given below:

- A point is that which has no part.
- A line is breadthless length.
- The ends of a line are points.
- A straight line is a line which lies evenly with the points on itself.
- A surface is that which has length and breadth only.
- The edges of a surface are lines.
- A plane surface is a surface which lies evenly with the straight lines on itself.

If you carefully study these definitions, you find that some of the terms like part, breadth, length, evenly, etc. need to be further explained clearly. *For example*, consider his definition of a point. In this definition, ‘a part’ needs to be defined. Suppose if you define ‘a part’ to be that which occupies ‘area’, again ‘an area’ needs to be defined. So, to define one thing, you need to define many other things, and you may get a long chain of definitions without an end. For such reasons, mathematicians agree to leave some geometric terms undefined. However, we do have an intuitive feeling for the geometric concept of a point than what the ‘definition’ above gives us. So, we represent a point as a dot, even though a dot has some dimension.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually ‘obvious universal truths’. He divided them into two types: axioms and postulates. He used the term ‘postulate’ for the assumptions that were specific to geometry. Common notions (often called axioms), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry.

Starting with his definitions, Euclid assumed certain properties, which were not to be proved. These assumptions are actually ‘obvious universal truths’. He divided them into two types: axioms and postulates. He used the term ‘postulate’ for the assumptions that were specific to geometry. Common notions (often called axioms), on the other hand, were assumptions used throughout mathematics and not specifically linked to geometry. For details about axioms and postulates,

Euclid’s axioms, not in his order, are given below:

- Things which are equal to the same thing are equal to one another.
- If equals are added to equals, the wholes are equal.
- If equals are subtracted from equals, the remainders are equal.
- Things which coincide with one another are equal to one another.
- The whole is greater than the part.
- Things which are double of the same things are equal to one another.
- Things which are halves of the same things are equal to one another.

Superposition is the ability of a quantum system to be in multiple states at the same time until it is measured.

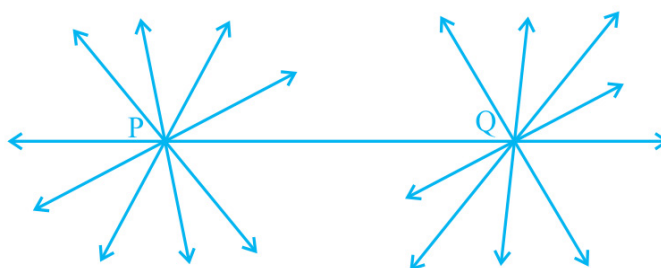
These ‘common notions’ refer to magnitudes of some kind. The first common notion could be applied to plane figures. *For example*, if an area of a triangle equals the area of a rectangle and the area of the rectangle equals that of a square, then the area of the triangle also equals the area of the square.

Magnitudes of the same kind can be compared and added, but magnitudes of different kinds cannot be compared. For example, a line cannot be added to a rectangle, nor can an angle be compared to a pentagon.

The 4th axiom given above seems to say that if two things are identical (that is, they are the same), then they are equal. In other words, everything equals itself. It is the justification of the principle of **superposition**. Axiom (5) gives us the definition of ‘greater than’. *For example*, if a quantity B is a part of another quantity A, then A can be written as the sum of B and some third quantity C. Symbolically, $A > B$ means that there is some C such that $A = B + C$.

Now let us discuss Euclid’s five postulates. They are:

- **Postulate 1:** A straight line may be drawn from any one point to any other point. Note that this postulate tells us that at least one straight line passes through two distinct points, but it does not say that there cannot be more than one such line. However, in his work, Euclid has frequently assumed, without mentioning, that there is a unique line joining two distinct points. We state this result in the form of an axiom as follows:
- **Axiom 1:** Given two distinct points, there is a unique line that passes through them. How many lines passing through P also pass through Q (see Figure)? Only one, that is, the line PQ. How many lines passing through Q also pass through P? Only one, that is, the line PQ. Thus, the statement above is self-evident, and so is taken as an axiom.

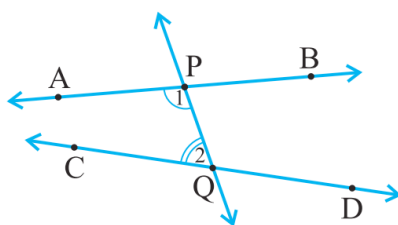


- *Postulate 2:* A terminated line can be produced indefinitely. Note that what we call a line segment now-a-days is what Euclid called a terminated line. So, according to the present day terms, the second postulate says that a line segment can be extended on either side to form a line (see Figure).



- *Postulate 3:* A circle can be drawn with any center and any radius.
- *Postulate 4:* All right angles are equal to one another.
- *Postulate 5:* If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of angles is less than two right angles.

For example, the line PQ in Figure falls on lines AB and CD such that the sum of the interior angles 1 and 2 is less than 180° on the left side of PQ. Therefore, the lines AB and CD will eventually intersect on the left side of PQ



A brief look at the five postulates brings to your notice that Postulate 5 is far more complex than any other postulate. On the other hand, Postulates 1 through 4 are so simple and obvious that these are taken as ‘self-evident truths’. However, it is not possible to prove them. So, these statements are accepted without any proof.

Now-a-days, ‘postulates’ and ‘axioms’ are terms that are used interchangeably and in the same sense. ‘Postulate’ is actually a verb. When we say “let us postulate”, we mean, “let us make some statement based on the observed phenomenon in the Universe”. Its truth/validity is checked afterwards. If it is true, then it is accepted as a ‘Postulate’.

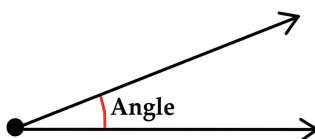
Angles

Once you have mastered the idea of points, lines and planes, the next thing to consider is what happens when two lines or rays meet at a point, creating an angle between them.

Angles are used throughout geometry, to describe shapes such as polygons and polyhedrons, and to explain the behavior of lines, so it’s a good idea to become familiar with some of the terminology, and how we measure and describe angles.

What is an Angle?

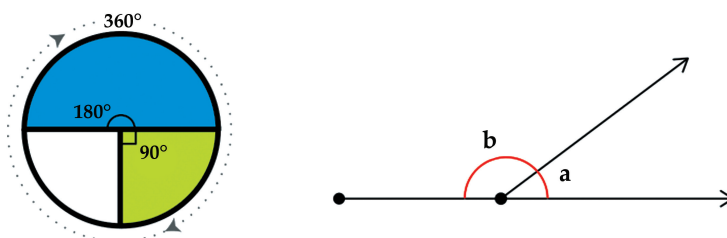
Angles are formed between two rays extending from a single point:



Angles are commonly drawn as an arc (part of a circle), as above.

Properties of Angles

Angles are measured in *degrees*, which is a measure of circularity, or rotation. A full rotation, which would bring you back to face in the same direction, is 360° . A half-circle is therefore 180° , and a quarter-circle, or right angle, is 90° .

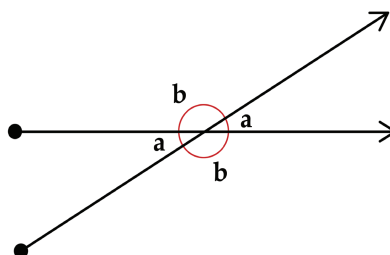


Two or more angles on a straight line add up to 180° . In the diagram above, the circle to the left is split into three sectors the angles of the green and white sectors are both 90° , adding up to 180° .

The figure to the right shows that angles a and b also add up to 180° . When you look at the diagram like this, it's easy to see this, but it's also surprisingly easy to forget in practice.

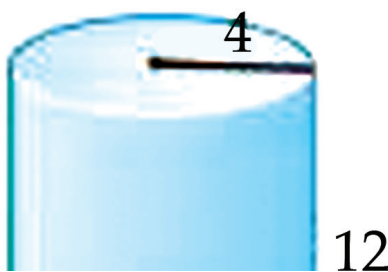
Opposite Angles: Intersecting Lines

When two lines intersect, the opposite angles are equal. In this case, not only are a and a the same, but, of course, a and b add up to 180° :



Intersections with parallel lines: a bit of a special case. The angles around any lines intersecting parallel lines also have some interesting properties.

If two parallel lines are intersected by a third straight line, then the angle at which the intersecting line crosses will be the same for both parallel lines.



The two angles a and the two angles b are said to be corresponding.

You will also immediately see that a and b add up to 180° , since they are on a straight line.

Angle c , which you will realize is identical to a , is said to be alternate with a .

Measuring Angles

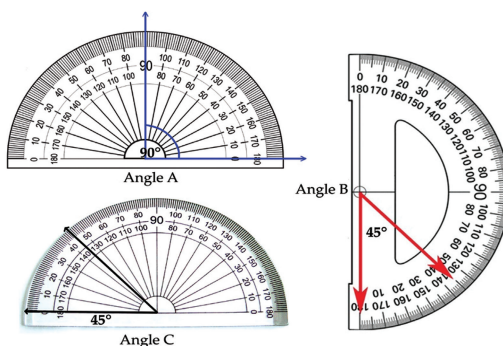
A protractor is commonly used to measure angles. Protractors are usually circular or semi-circular and made of transparent plastic, so that they can be placed over shapes drawn on a piece of paper, allowing you to take a measurement of the angle.

This example demonstrates how to use a protractor to measure the three angles of a triangle, but the same method applies to other shapes or any angles that you want to measure.

Line up the central mark on the base of your protractor with the vertex, or point at which the lines meet. The triangle has three vertices, one for each angle to measure.

Most protractors have a bi-directional scale meaning that you can take a measurement in either direction. Make sure you use the correct scale – you should be able to tell easily if your angle is greater than or less than 90° and therefore use the right scale. If you are not sure, take a quick look back to our section on naming angles.

Angle is used to designate the measure of an angle or of a rotation. This measure is the ratio of the length of a circular arc to its radius. In the case of a geometric angle, the arc is centered at the vertex and delimited by the sides. In the case of a rotation, the arc is centered at the center of the rotation and delimited by any other point and its image by the rotation.



In this example the recorded angles are $A=90^\circ$ $B=45^\circ$ and $C=45^\circ$.

Polygon is a blockchain platform which aims to create a multi-chain blockchain system compatible with Ethereum.

Polygons are often defined by their internal angles, and the total of the internal angles depends on the number of sides. For example, the internal angles of a triangle always add up to 180° . For more about this, see our page on Polygons.

Polygon

In elementary geometry, a polygon is a plane figure that is bounded by a finite chain of straight line segments closing in a loop to form a closed polygonal chain or circuit. These segments are called its edges or sides, and the points where two edges meet are the **polygon's** vertices (singular: vertex) or corners. The interior of the polygon is sometimes called its body. An n-gon is a polygon with n sides; for example, a triangle is a 3-gon. A polygon is a 2-dimensional example of the more general polytope in any number of dimensions.

The basic geometrical notion of a polygon has been adapted in various ways to suit particular purposes. Mathematicians are often concerned only with the bounding closed polygonal chain and with simple polygons which do not self-intersect, and they often define a polygon accordingly. A polygonal boundary may be allowed to intersect itself, creating star polygons and other self-intersecting polygons.

Properties of Polygons

The properties of two-dimensional or 'plane' polygons. A polygon is any shape made up of straight lines that can be drawn on a flat surface, like a piece of paper. Such shapes include squares, rectangles, triangles and pentagons but not circles or any other shape that includes a curve.

Understanding shapes is important in mathematics. You will certainly be required to learn about shapes at school but understanding the properties of shapes has many practical applications in professional and real-life situations too.

Number of Sides

Polygons are often defined by the number of sides that they have.

Triangles – Three Sides

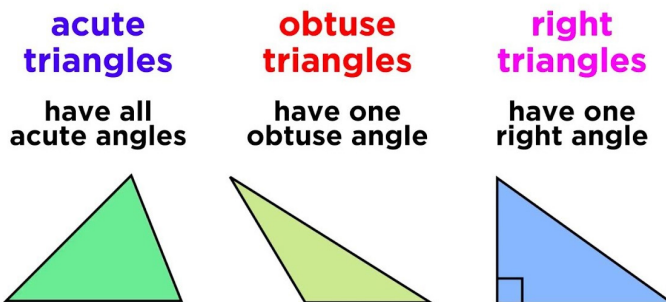
A three-sided polygon is a triangle. There are several different types of triangle (see diagram), including:

- Equilateral – all the sides are equal lengths, and all the internal angles are 60° .
- Isosceles – has two equal sides, with the third one a different length. Two of the internal angles are equal.
- Scalene – all three sides, and all three internal angles, are different.

Triangles can also be described in terms of their internal angles.

A triangle with only acute internal angles is called an **acute triangle**. One with an obtuse angle is called obtuse, and one with a right angle is known as right angled.

Each of these will *also* be either equilateral, isosceles or scalene.



Acute triangle is a triangle with all three angles acute (less than 90°). An obtuse triangle is one with one obtuse angle (greater than 90°) and two acute angles. Since a triangle's angles must sum to 180° , no triangle can have more than one obtuse angle.

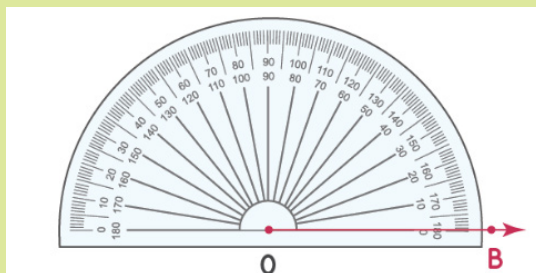
Axioms

An axiom or postulate is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments. The word comes from the Greek 'that which is thought worthy or fit' or 'that which commends itself as evident. The term has subtle differences in definition when used in the context of different fields of study. As defined in classic philosophy, an axiom is a statement that is so evident or well-established, that it is accepted without controversy or question. As used in modern logic, an axiom is simply a premise or starting point for reasoning.

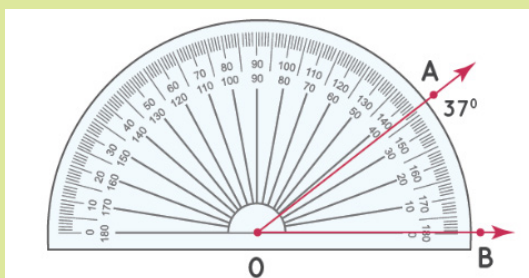
CONCEPT: HOW TO MEASURE AN ACUTE ANGLE?

Let us try to measure the given $\angle AOB$.

Step 1: Align the protractor with the ray OB as shown below. Start reading the inner scale from the 0° mark on the bottom-right of the protractor.



Step 2: The number on the protractor that coincides with the second ray is the measure of the angle. Measure the angle using the inner scale of the protractor. Thus, $\angle AOB = 37^\circ$



SUMMARY

- Geometry is a fundamental branch of mathematics that explores the properties, relationships, and configurations of shapes, sizes, and spaces. It plays a crucial role in understanding the physical world and has applications in various fields, from architecture and engineering to art and physics.
- Geometry is one of the oldest branches of mathematics. The word geometry in the Greek language translates the words for “Earth” and “Measure”. The Egyptians were one of the first civilizations to use geometry. The Egyptians used right triangles to measure and survey land.
- A ray is a cross between a line and a line segment. It extends without bound in one direction, but not the other. It is determined by two points, one being the starting point for the ray, and the other determining the direction of the ray.
- A curve is a 1-dimensional object that may be straight (like a line) or not; curves in 2-dimensional space are called plane curves and those in 3-dimensional space are called space curves.
- Angles are used throughout geometry, to describe shapes such as polygons and polyhedrons, and to explain the behavior of lines, so it’s a good idea to become familiar with some of the terminology, and how we measure and describe angles.
- A protractor is commonly used to measure angles. Protractors are usually circular or semi-circular and made of transparent plastic, so that they can be placed over shapes drawn on a piece of paper, allowing you to take a measurement of the angle.
- A polygon is a plane figure that is bounded by a finite chain of straight line segments closing in a loop to form a closed polygonal chain or circuit. These segments are called its edges or sides, and the points where two edges meet are the polygon’s vertices (singular: vertex) or corners.

MULTIPLE CHOICE QUESTIONS

1. **What is the sum of the interior angles of a triangle?**
 - a. 90 degrees
 - b. 180 degrees
 - c. 270 degrees
 - d. 360 degrees
2. **Which of the following shapes has no sides of equal length and no angles of equal measure?**
 - a. Square
 - b. Rectangle
 - c. Equilateral triangle
 - d. Scalene triangle
3. **What is the name of a three-dimensional shape with six rectangular faces, twelve edges, and eight vertices?**
 - a. Sphere
 - b. Cylinder
 - c. Cube
 - d. Cone
4. **In a right triangle, which side is opposite the right angle?**
 - a. Hypotenuse
 - b. Adjacent side
 - c. Opposite side
 - d. None of the above
5. **Two angles whose measures add up to 180 degrees are called:**
 - a. Complementary angles
 - b. Supplementary angles
 - c. Adjacent angles
 - d. Vertical angles

Answer

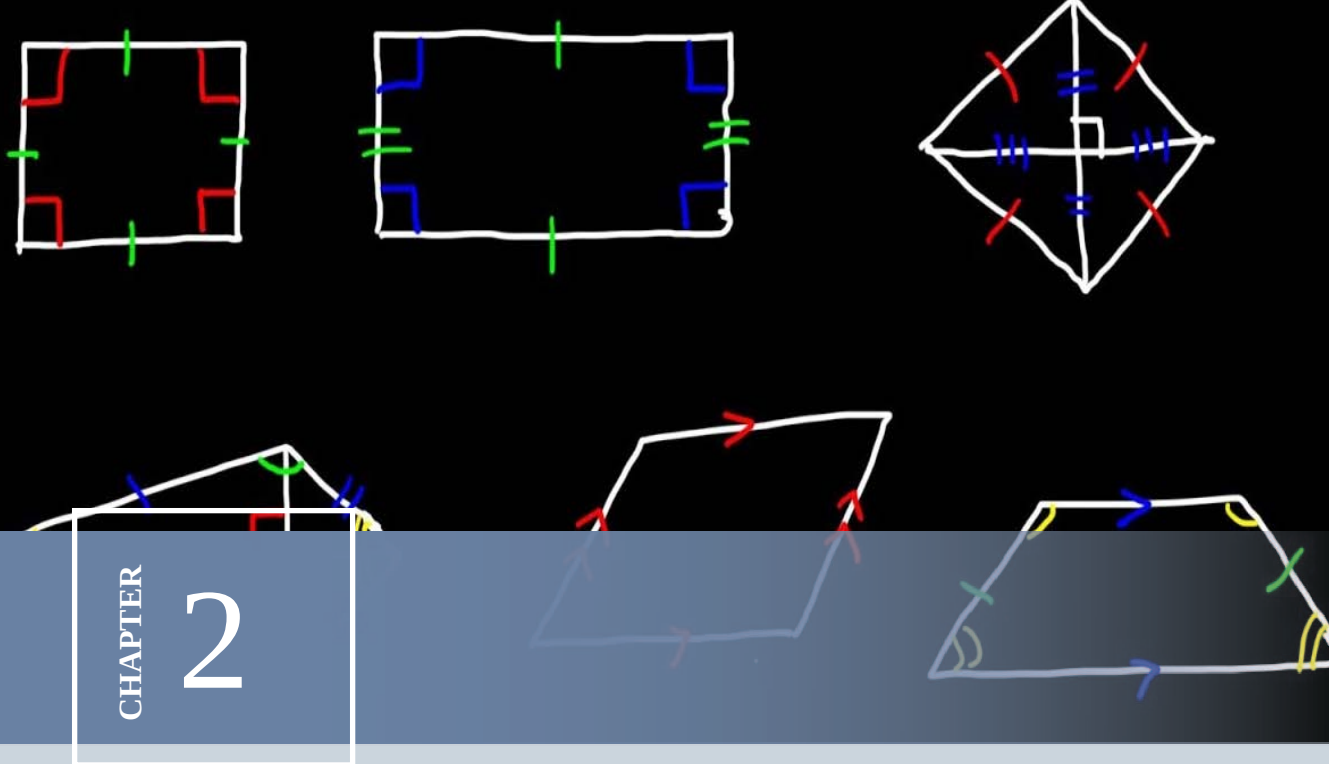
1. (b) 2. (d) 3. (c) 4. (a) 5. (b)

REVIEW QUESTIONS

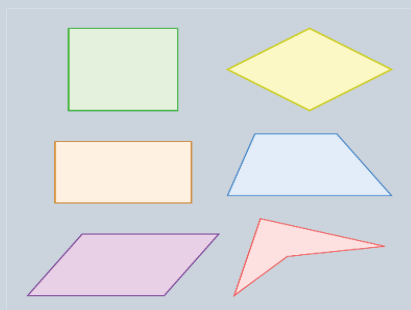
1. Discuss about important concepts in geometry.
2. Focus on parallel and perpendicular lines.
3. Explain the planes and two-dimensional shapes.
4. What is an angle? Explain properties of an angles.
5. What is polygon? Write its properties.

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QUADRILATERALS



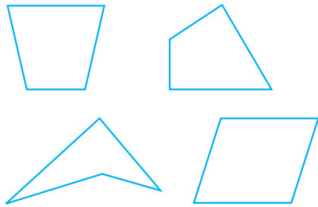
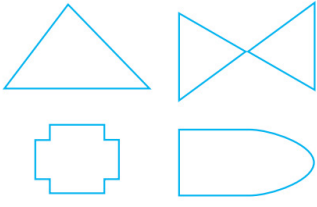
Quadrilaterals are 4-sided polygons. Among them those whose vertices lie on a circle are called cyclic quadrilaterals. Cyclic quadrilaterals are the simplest objects like triangles in plane geometry and they possess remarkable properties.

After studying this chapter, you will understand the core concepts of:

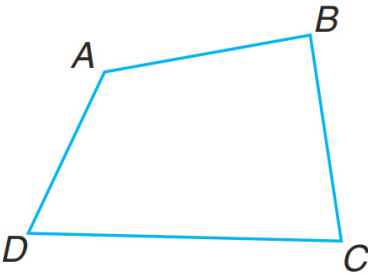
- Overview of Quadrilaterals
- Ptolemy's Theorem
- Pedal Triangle

INTRODUCTION

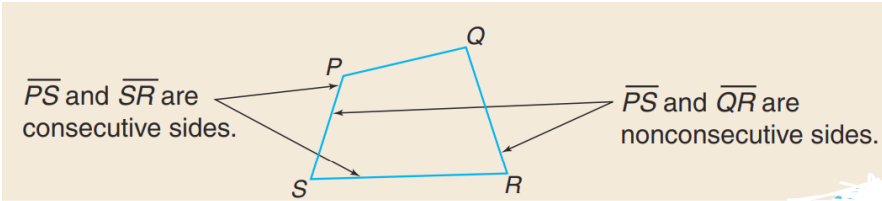
A quadrilateral is a closed geometric figure with four sides and four vertices. The segments of a quadrilateral intersect only at their endpoints. Special types of quadrilaterals include squares and rectangles.

Quadrilaterals	Not Quadrilaterals
	

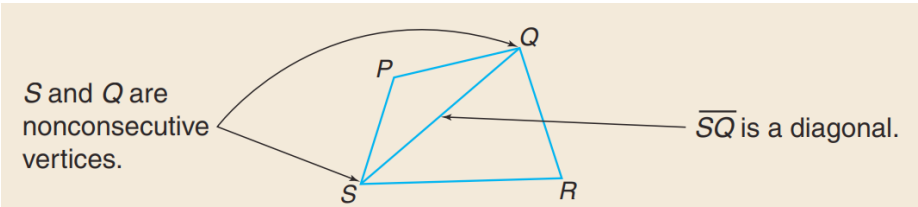
Quadrilaterals are named by listing their vertices in order. There are many names for the quadrilateral at the right. Some examples are quadrilateral ABCD, quadrilateral BCDA, or quadrilateral DCBA.

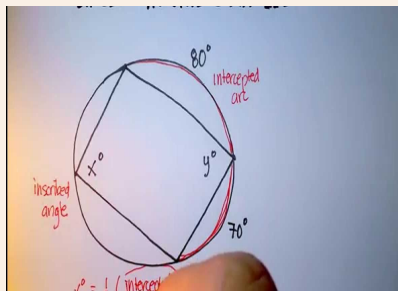


Any two sides, vertices, or angles of a quadrilateral are either consecutive or nonconsecutive.



Segments that join nonconsecutive vertices of a quadrilateral are called diagonals.



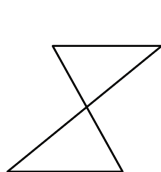


CORE CONCEPT: OVERVIEW OF QUADRILATERALS

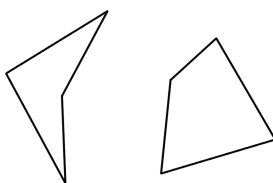
A Quadrilateral has four-sides, it is 2-dimensional (a flat shape), closed (the lines join up), and has straight sides. Sometimes, the term quadrangle is used, by analogy with triangle, and sometimes tetragon for consistency with pentagon (5-sided), hexagon (6-sided) and so on. The interior angles of a simple (and planar) quadrilateral ABCD add up to 360 degrees of arc, that is

$\angle A + \angle B + \angle C + \angle D = 360^\circ$

A quadrilateral is a figure with four sides; common examples of quadrilaterals include squares, rectangles, and trapezoids. Quadrilaterals break down into several categories: simple quadrilaterals are those that do not have any self-crossings (in the manner of the bowtie shape below, for example); complex quadrilaterals do have self-crossings.



Complex quadrilateral



Simple quadrilaterals

In the class of simple quadrilaterals are concave quadrilaterals (such as the leftmost simple quadrilateral above) and convex quadrilaterals. A concave quadrilateral contains a reflex angle (an angle greater than 180°), whereas all of the angles in a convex quadrilateral are less than 180° . Again, the examples in the diagram above illustrate these characteristics. In our discussion, we will only consider simple convex quadrilaterals--these include such figures as rectangles. In addition to having four sides, quadrilaterals also have four vertices and four interior angles. One particular property of quadrilaterals that we can immediately derive is the total number of degrees in the sum of interior angles. Recall that our study of triangles showed that every triangle has 180° (that is, the measures of the interior angles always sum to 180°). We can use this fact about triangles to determine the sum of the angles in a quadrilateral. Consider the generic quadrilateral below; we can draw a **diagonal** line between two non-adjacent vertices to form two triangles.

Diagonal is a line segment joining two vertices of a polygon or polyhedron, when those vertices are not on the same edge. Informally, any sloping line is called diagonal.



Thus, if we add up all the measures of the interior angles in the two triangles, we then know how many degrees are in a quadrilateral. Of course, this sum is twice 180° , or a total of 360° . This result makes sense if you already know that a rectangle contains four right angles.

Types of Quadrilaterals

There are five types of quadrilaterals.

1. Parallelogram
2. Rectangle
3. Square
4. Rhombus
5. Trapezium

Parallelogram

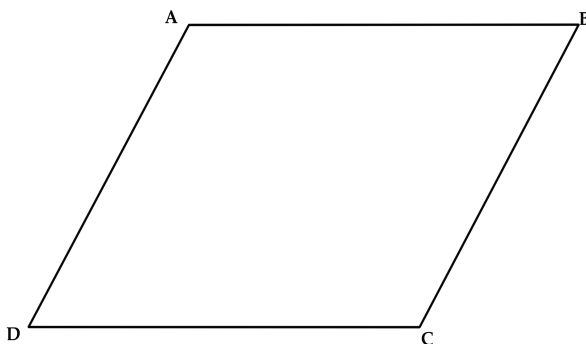
A parallelogram is a quadrilateral with opposite sides parallel. But there are various tests that can be applied to see if something is a parallelogram. If a quadrilateral has two pairs of parallel opposite sides, then it is called a parallelogram. Parallel sides in a parallelogram are also congruent in length. In addition, because opposite sides are parallel, opposite angles are congruent as well.

It is the “parent” of some other quadrilaterals, which are obtained by adding restrictions of various kinds:

- A rectangle is a parallelogram but with all four interior angles fixed at 90°
- A rhombus is a parallelogram but with all four sides equal in length
- A square is a parallelogram but with all sides equal in length and all interior angles 90°

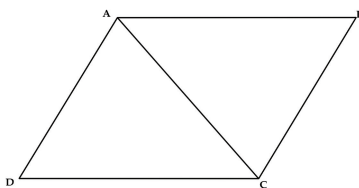
Properties of parallelograms

One special kind of polygons is called a parallelogram. It is a quadrilateral where both pairs of opposite sides are parallel.



There are six important properties of parallelograms to know:

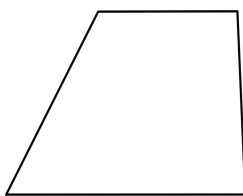
1. Opposite sides are congruent ($AB = DC$).
2. Opposite angles are congruent ($D = B$).
3. Consecutive angles are supplementary ($A + D = 180^\circ$).
4. If one angle is right, then all angles are right.
5. The diagonals of a parallelogram bisect each other.
6. Each diagonal of a parallelogram separates it into two congruent triangles.



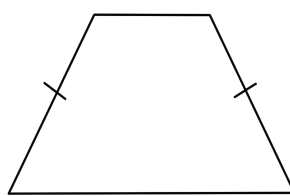
$$\triangle ACD \cong \triangle ABC$$

If we have a parallelogram where all sides are congruent then we have what is called a rhombus. The properties of parallelograms can be applied on rhombi.

If we have a quadrilateral where one pair and only one pair of sides are parallel then we have what is called a trapezoid. The parallel sides are called bases while the nonparallel sides are called legs. If the legs are congruent we have what is called an isosceles trapezoid.



Trapezoid

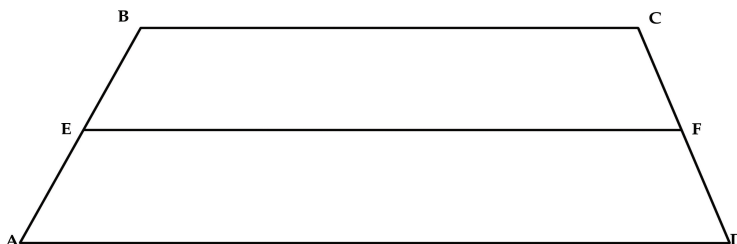


Isosceles trapezoid

Isosceles trapezoid

is a convex quadrilateral with a line of symmetry bisecting one pair of opposite sides. It is a special case of a trapezoid.

In an **isosceles trapezoid** the diagonals are always congruent. The median of a trapezoid is parallel to the bases and is one-half of the sum of measures of the bases.



$$EF = \frac{1}{2}(AD + BC)$$

Rectangle

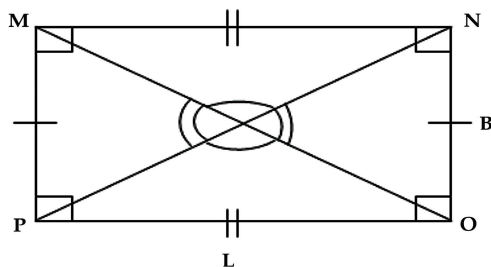
A rectangle is a quadrilateral with four right angles. It can also be defined as an equiangular quadrilateral, since equiangular means that all of its angles are equal ($360^\circ/4 = 90^\circ$). It can also be defined as a parallelogram containing a right angle. A rectangle with four sides of equal length is a square. The term oblong is occasionally used to refer to a non-square rectangle.

A crossed rectangle is a crossed (self-intersecting) quadrilateral which consists of two opposite sides of a rectangle along with the two diagonals. It is a special case of an antiparallelogram, and its angles are not right angles. Other geometries, such as spherical, elliptic, and hyperbolic, have so-called rectangles with opposite sides equal in length and equal angles that are not right angles. Rectangles are involved in many tiling problems, such as tiling the plane by rectangles or tiling a rectangle by polygons.

Properties of a Rectangle

- Opposite sides are parallel and congruent.
- All angles are right.
- The diagonals are congruent and bisect each other (divide each other equally).
- Opposite angles formed at the point where diagonals meet are congruent.
- A rectangle is a special type of parallelogram whose angles are right.

Rectangles



Important Formulas for Rectangles

- If the length is L and breadth is B, then
Length of the diagonal of a rectangle = $\sqrt{L^2 + B^2}$
- Area = $L * B$
- Perimeter = $2(L+B)$

Characterizations for Rectangles

A convex quadrilateral is a rectangle if and only if it is any one of the following:

- a parallelogram with at least one right angle
- a parallelogram with diagonals of equal length
- a parallelogram ABCD where triangles ABD and DCA are congruent
- an equiangular quadrilateral
- a quadrilateral with four right angles
- a convex quadrilateral with successive sides a, b, c, d whose area is $\frac{1}{4}(a+c)(b+d)$
- a convex quadrilateral with successive sides a, b, c, d whose area is $\frac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}$

Square

A Square is a flat shape with 4 equal sides and every angle is a right angle (90°). A square is a regular quadrilateral, which means that it has four equal sides and four equal angles (90-degree angles, or (100-gradian angles or right angles). It can also be defined as a rectangle in which two adjacent sides have equal length.

Characterizations

A convex quadrilateral is a square if and only if it is any one of the following:

- a rectangle with two adjacent equal sides
- a rhombus with a right vertex angle

A **rhombus** is a quadrilateral with both pairs of opposite sides parallel and all sides the same length, i.e., an equilateral parallelogram.

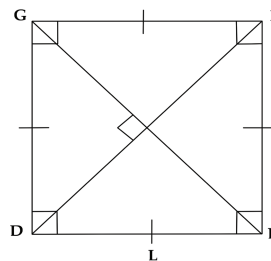
- a rhombus with all angles equal
- a parallelogram with one right vertex angle and two adjacent equal sides
- a quadrilateral with four equal sides and four right angles
- a quadrilateral where the diagonals are equal and are the perpendicular bisectors of each other, i.e. a **rhombus** with equal diagonals
- a convex quadrilateral with successive sides a, b, c,

$$d \text{ whose area is } A = \frac{1}{2}(a^2 + c^2) = \frac{1}{2}(b^2 + d^2)$$

Properties of a Square

- All sides and angles are congruent.
- Opposite sides are parallel to each other.
- The diagonals are congruent.
- The diagonals are perpendicular to and bisect each other.
- A square is a special type of parallelogram whose all angles and sides are equal.
- Also, a parallelogram becomes a square when the diagonals are equal and right bisectors of each other.

Squares



Important formulas for Squares

- If 'L' is the length of the side of a square then length of the diagonal = $L\sqrt{2}$.
- Area = L^2 .
- Perimeter = $4L$

Rhombus

A rhombus is a simple quadrilateral whose four sides all have the same length. Another name is equilateral quadrilateral, since

equilateral means that all of its sides are equal in length. The rhombus is often called a diamond, after the diamonds suit in playing cards which resembles the projection of an octahedral diamond, or a lozenge, though the former sometimes refers specifically to a rhombus with a 60° angle (see Polyiamond), and the latter sometimes refers specifically to a rhombus with a 45° angle.

A rhombus is a type of parallelogram, and what distinguishes its shape is that all four of its sides are congruent.

There are several formulas for the rhombus that have to do with its

- Sides
all 4 sides are congruent
- Angles
diagonals bisect vertex angles
- Diagonals
diagonals are perpendicular
- Area

A square is a rhombus and a rectangle. In other words, if each angle of a rhombus is 90° , then it's a square.

Characterizations for Rhombus

A simple (non-self-intersecting) quadrilateral is a rhombus if and only if it is any one of the following:

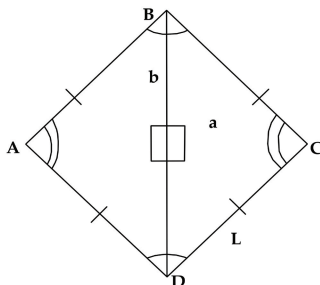
- a parallelogram in which a diagonal bisects an interior angle
- a parallelogram in which at least two consecutive sides are equal in length
- a parallelogram in which the diagonals are perpendicular (an orthodiagonal parallelogram)
- a quadrilateral with four sides of equal length
- a quadrilateral in which the diagonals are perpendicular and bisect each other
- a quadrilateral in which each diagonal bisects two opposite interior angles
- a quadrilateral ABCD possessing a point P in its plane such that the four triangles ABP, BCP, CDP, and DAP are all congruent
- a quadrilateral ABCD in which the incircles in triangles ABC, BCD, CDA and DAB have a common point

Properties of a Rhombus

- All sides are congruent.
- Opposite angles are congruent.
- The diagonals are perpendicular to and bisect each other.
- Adjacent angles are supplementary (For eg., $\angle A + \angle B = 180^\circ$).

- A rhombus is a parallelogram whose diagonals are perpendicular to each other.

Rhombus



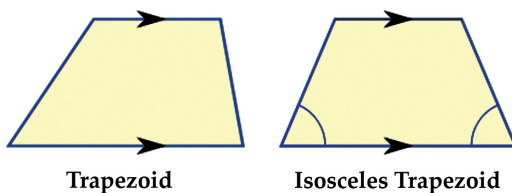
Important formulas for a Rhombus

If a and b are the lengths of the diagonals of a rhombus,

- Area = $(a * b) / 2$
- Perimeter = $4L$

Trapezium

A trapezoid is a 4-sided flat shape with straight sides that has a pair of opposite sides parallel. A trapezium is defined by the properties it does not have. It has no parallel sides. Any quadrilateral drawn at random would probably be a trapezium. Since it has no interesting properties beyond those of a quadrilateral, it is not used much in geometry. If the quadrilateral had one pair of parallel sides, it would be a trapezoid. If both pairs of sides are parallel, it would be a parallelogram.



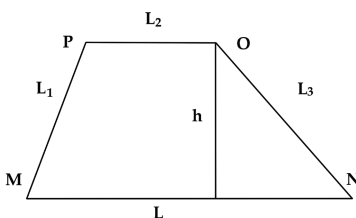
Trapezoid

Isosceles Trapezoid

Properties of a Trapezium

- The bases of the trapezium are parallel to each other (MN $\parallel$ OP).
- No sides, angles and diagonals are congruent.

Trapezium

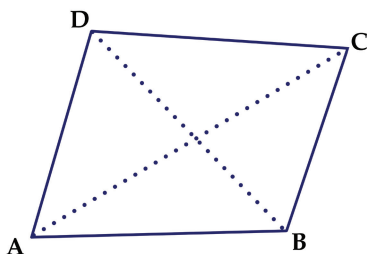


Important Formulas for a Trapezium

- Area = $(1/2) h (L + L_2)$
- Perimeter = $L + L_1 + L_2 + L_3$

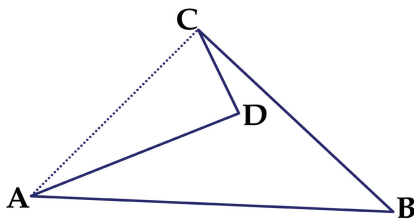
Convex and Concave Quadrilaterals

A quadrilateral is called a convex quadrilateral, if the line segment joining any two vertices of the quadrilateral is in the same region. In figure, ABCD is a convex quadrilateral because AB, BC, CD, DA, AC, BD are in the same region of the quadrilateral.



In a convex quadrilateral each angle measures less than 180° .

A quadrilateral is called a concave quadrilateral, if at least one line segment joining the vertices is not a part of the same region of the quadrilateral. That is, any line segment that joins two interior points goes outside the figure. In a concave quadrilateral at least one angle is a **reflex angle**, i.e., an angle larger than 180° . In figure, ABCD is a concave quadrilateral because a line joining the vertices A and C is going outside the quadrilateral region.

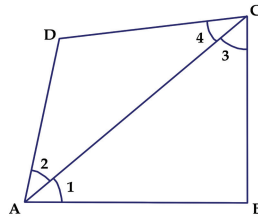


Reflex angle is the larger angle. It is more than 180° but less than 360° . If you choose the smaller angle you might have an Acute Angle, or an Obtuse Angle instead. The larger angle is a Reflex Angle, but the smaller angle is an Acute Angle.

Angle Sum Property of a Quadrilateral

The sum of all angles of a quadrilateral is 360° or four right angles.

Draw a quadrilateral ABCD with one of its diagonals AC.



Diagonal AC divides the quadrilateral into two triangles, i.e., $\triangle ADC$ and $\triangle ABC$.

Clearly $\angle 1 + \angle 2 = \angle A$

and $\angle 3 + \angle 4 = \angle C$ (1)

We know that the sum of the angles of a triangle is 180° .

$\therefore$ In $\triangle ABC$, $\angle 1 + \angle 3 + \angle B = 180^\circ$

In $\triangle ADC$, $\angle 2 + \angle 4 + \angle D = 180^\circ$

Sum of the angles of a quadrilateral

= Sum of the angles of $\triangle ABC$ and $\triangle ADC$

$\therefore (\angle 1 + \angle 3 + \angle B) + (\angle 2 + \angle 4 + \angle D) = 180^\circ + 180^\circ$

or $\angle 1 + \angle 3 + \angle B + \angle 2 + \angle 4 + \angle D = 360^\circ$

or $(\angle 1 + \angle 2) + \angle B + (\angle 3 + \angle 4) + \angle D = 360^\circ$

or $\angle A + \angle B + \angle C + \angle D = 360^\circ$ (using 1)

Hence, the sum of the angles of a quadrilateral equals 360° .

Example 1: The angles of a quadrilateral are in the ratio of 1 : 2 : 1 : 2. Find the measure of each angle.

Solution: Let the first angle of a quadrilateral be x .

Here, second angle = $2x$

third angle = x

fourth angle = $2x$

Sum of all angles of a quadrilateral = 360°

$\therefore x + 2x + x + 2x = 360^\circ$

$6x = 360^\circ$

$x = 60^\circ$

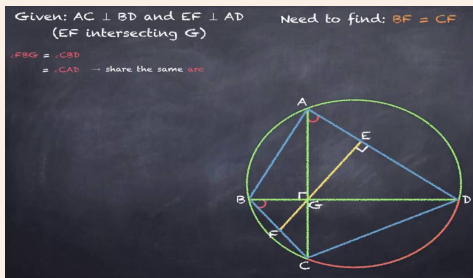
$\therefore$ First angle = $x = 60^\circ$

Second angle = $2x = 2 \times 60^\circ = 120^\circ$

Third angle = $x = 60^\circ$

Fourth angle = $2x = 120^\circ$.

Quadrilaterals are grouped by their angles, by the lengths of their sides, and by any parallel sides.



CORE CONCEPT: PTOLEMY'S THEOREM

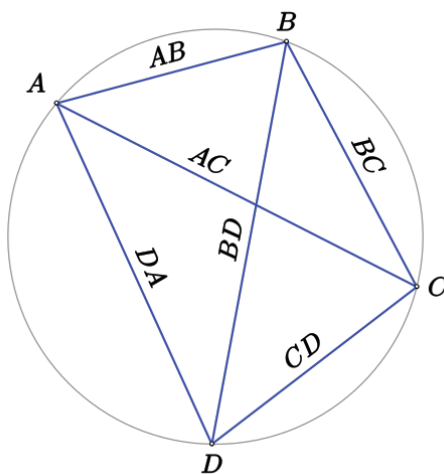
Ptolemy's Theorem gives a relationship between the side lengths and the diagonals of a cyclic quadrilateral; it is the equality case of Ptolemy's Inequality. Ptolemy's Theorem is a relation in Euclidean geometry between the four sides and two diagonals of a cyclic quadrilateral (i.e., a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy (Claudius Ptolemaeus).

Ptolemy's Theorem states that, in a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. Let a quadrilateral $ABCD$ be inscribed in a circle. Then prove the sum of the products of the two pairs of opposite sides equals the product of its two diagonals. In other words,

$$AD \cdot BC + AB \cdot CD = AC \cdot BD$$

There are many approaches to a proof of this important traditional geometry theorem. One approach is to construct an auxiliary figure to achieve similar triangles which can then be used to set up the desired result. The following is one way to approach that. That angles CDB and CAB subtend the same chord CB . Therefore the angles have the same measure.

- Locate point M on BD such that $\angle ACB = \angle MCD$.
- This will allow the establishment of similar triangles DMC and ABC .
- Likewise we can establish similar triangles CAD and CBM .



Circumcircle is a triangle's circumscribed circle, i.e., the unique circle that passes through each of the triangle's three vertices. The center of the circumcircle is called the circumcenter, and the circle's radius is called the circumradius.

Theorem (The Simson line)

The feet of the perpendiculars from any point P on the **circumcircle** of a triangle ABC to the sides of the triangle are collinear.

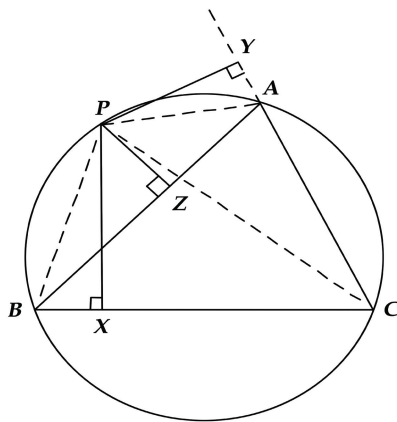


Figure 1: The Simson line.

Proof

Referring to figure 1, we see that $PZAY$, $PXCZ$ and $PACB$ are cyclic quadrilaterals. Therefore, $\angle PYZ = \angle PAZ = \angle PCX = \angle PZX$. This shows that Y, Z, X are collinear.

(Note that the converse of the statement in this theorem is also true. That is, if the feet of the perpendiculars from a point P to the sides of the triangle ABC are collinear, then P lies on the circumcircle of $\triangle ABC$.) The line containing the feet is known as the Simson line.

Theorem (Ptolemy)

For any cyclic quadrilateral, the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.

Proof

Let $PBCA$ be a cyclic quadrilateral and let X, Y, Z be the feet of the perpendiculars from P onto the sides BC, AC, AB respectively. By previous theorem, X, Y, Z lie on the Simson line.

The quadrilateral $AYPZ$ is cyclic. Since $\angle PYZ = 90^\circ$, the circle passing through A, Y, P, Z has diameter PA . Thus

$$\frac{YZ}{PA} = \sin \angle YAZ = \sin \angle BAC = \frac{a}{2R}$$

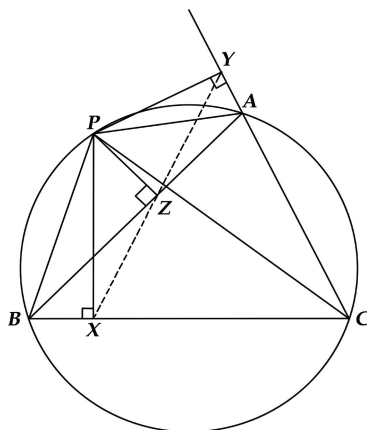


Figure 2: Ptolemy's theorem.

That is $YZ = aPA/(2R)$. Similarly, by considering the cyclic quadrilaterals $PZXB$ and $PXCX$, we have $XZ = bPB/(2R)$ and $XY = cPC/(2R)$. As X, Y, Z lie on the Simson line, we have $XZ + ZY = XY$ so that $bPB/(2R) + aPA/(2R) = cPC/(2R)$. Canceling the common factor $2R$, we get $bPB + aPA = cPC$. That is

$$AC \cdot PB + BC \cdot PA = AB \cdot PC$$

Ptolemy's Theorem can be strengthened by observing that if P is any point not on the circumcircle of $\triangle ABC$, then the equality $XZ + ZY = XY$ has to be replaced by the inequality $XZ + ZY > XY$ so that $AC \cdot PB + BC \cdot PA > AB \cdot PC$.

Theorem

If P is a point not on the arc CA of the circum-circle of the triangle ABC , then

$$AC \cdot PB + BC \cdot PA > AB \cdot PC$$

Example

Let P be a point of the minor arc CD of the circum-circle of a square $ABCD$. Prove that

$$PA(PA + PC) = PB(PB + PD)$$

Solution

Refer to figure 3. Let $AB = a$. Applying Ptolemy's theorem to the cyclic quadrilaterals $PDAB$ and $PABC$, we have $PD \cdot BA + PB \cdot DA = PA \cdot DB$, and $PA \cdot BC + PC \cdot AB = PB \cdot AC$. That is $a(PD + PB) = \sqrt{2}a \cdot PA$ and $a(PA + PC) = \sqrt{2}a \cdot PB$. Canceling

a common factor of a for both equations, we get $PD + PB = \sqrt{2} PA$ and $PA + PC = \sqrt{2} PB$. Thus $PA(PA + PC) = \sqrt{2} PA \cdot PB = PB(PB + PD)$.

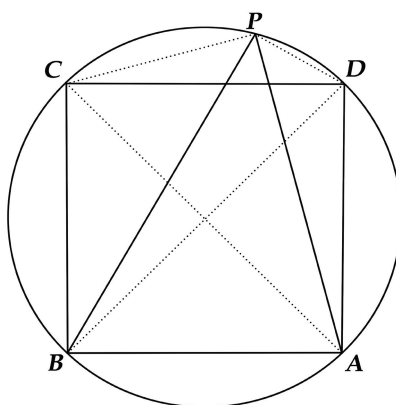


Figure 3: Applying Ptolemy's theorem.

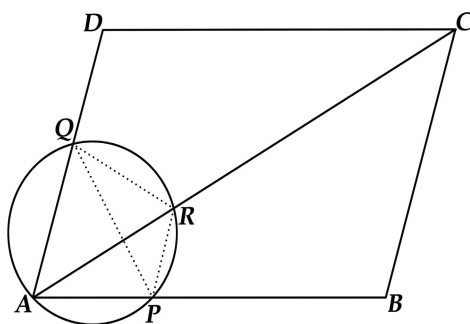


Figure 4: Parallelogram.

Exercise 1

In a parallelogram ABCD, a circle passing through A meets AB, AD and AC at P, Q and R respectively. Prove that $AP \cdot AB + AQ \cdot AD = AR \cdot AC$. See figure 4.

Generalizations of the Parallelogram Law and Ptolemy's Theorem

In any convex quadrilateral ABCD, the sum of the squares of the four sides is equal to the sum of the squares of the two diagonals plus four times the square of the line segment connecting the midpoints of the diagonals. Thus

$$a^2 + b^2 + c^2 + d^2 = p^2 + q^2 + 4x^2$$

where x is the distance between the midpoints of the diagonals. This is sometimes known as Euler's quadrilateral theorem and is a generalization of the parallelogram law.

The German mathematician Carl Anton Bretschneider derived in 1842 the following generalization of Ptolemy's theorem, regarding the product of the diagonals in a convex quadrilateral

$$p^2 q^2 = a^2 c^2 + b^2 d^2 - 2abcd \cos(A + C)$$

This relation can be considered to be a law of cosines for a quadrilateral. In a cyclic quadrilateral, where $A + C = 180^\circ$, it reduces to $pq = ac + bd$. Since $\cos(A + C) \geq -1$, it also gives a proof of Ptolemy's inequality.

If a quadrilateral is inscribable in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides.

Other Metric Relations

If X and Y are the feet of the normals from B and D to the diagonal AC = p in a convex quadrilateral ABCD with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, then

$$XY = \frac{|a^2 + c^2 - b^2 - d^2|}{2p}$$

In a convex quadrilateral ABCD with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$, and where the diagonals intersect at E,

$$efgh(a + c + b + d)(a + c - b - d) = (agh + cef + beh + dfg)(agh + cef - beh - dfg)$$

where $e = AE$, $f = BE$, $g = CE$, and $h = DE$.

The shape and size of a convex quadrilateral are fully determined by the lengths of its sides in sequence and of one diagonal between two specified vertices. The two diagonals p, q and the four side lengths a, b, c, d of a quadrilateral are related by the Cayley-Menger determinant, as follows:

$$\det \begin{bmatrix} 0 & a^2 & p^2 & d^2 & 1 \\ a^2 & 0 & b^2 & q^2 & 1 \\ p^2 & b^2 & 0 & c^2 & 1 \\ d^2 & q^2 & c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = 0.$$

Area of a Quadrilateral

Quadrilateral is a closed convex polygon bounded by four sides. We have already seen that the quadrilaterals are classified according to the interior angles and the sides. Therefore, the names of the quadrilaterals vary according to their properties. Areas of rectangle, square, parallelograms can be found by the suitable formulas derived. If we are given a quadrilateral whose sides or angles are unequal, we can find the areas by splitting it into triangles which do not overlap.

Theorem (Brahmagupta's Formula)

If a cyclic quadrilateral has sides a, b, c, d and semiperimeter s , then its area K is given by

$$K^2 = (s - a)(s - b)(s - c)(s - d)$$

Proof

Let $ABCD$ be a cyclic quadrilateral. Let the length of BD be n . First note that $\angle A + \angle C = 180^\circ$ so that $\cos A = -\cos C$ and $\sin A = \sin C$. Thus by Cosine law

$$a^2 + b^2 - 2ab \cos A = n^2 = c^2 + d^2 - 2cd \cos C,$$

giving

$$2(ab + cd) \cos A = a^2 + b^2 - c^2 - d^2. \quad (2)$$

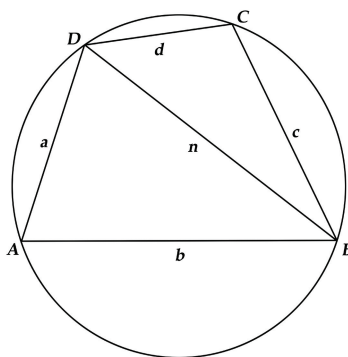


Figure 5: Brahmagupta's Formula.

Since

$$K = \frac{1}{2}ab \sin A + \frac{1}{2}cd \sin C = \frac{1}{2}(ab + cd) \sin A,$$

$$2(ab + cd) \sin A = 4K. \quad (3)$$

we also have

Adding the squares of (2) and (3), we obtain

$$4(ab + cd)^2 = (a^2 + b^2 - c^2 - d^2)^2 + 16K^2$$

giving

$$16K^2 = (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2.$$

Thus

$$16K^2 = (2ab + 2cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$$

$$= (2ab + 2cd + a^2 + b^2 - c^2 - d^2)(2ab + 2cd - a^2 - b^2 + c^2 + d^2)$$

$$\begin{aligned}
&= ((a+b)^2 - (c-d)^2)((c+d)^2 - (a-b)^2) \\
&= (a+b+c-d)(a+b-c+d)(c+d+a-b)(c+d-a+b) \\
&= (2s-2d)(2s-2c)(2s-2b)(2s-2a).
\end{aligned}$$

Therefore,

$$K^2 = (s-a)(s-b)(s-c)(s-d)$$

Setting $d = 0$, we obtain Heron's formula for the area of a triangle:

$$(ABC)^2 = s(s-a)(s-b)(s-c).$$

Exercise 2

In a trapezium ABCD, AB is parallel to DC and E is the midpoint of BC. Prove that $2(AED) = (ABCD)$.

Exercise 3

Suppose the quadrilateral ABCD has an inscribed circle. Show that $AB + CD = BC + DA$.

Exercise 4

Suppose the cyclic quadrilateral ABCD has an inscribed circle. Show that $(ABCD) = \sqrt{abcd}$.

Exercise 5

Let ABCD be a convex quadrilateral. Prove that its area K is given by

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \left(\frac{A+C}{2} \right)$$

Exercise 6

Let ABCDE be the pentagon whose vertices are intersections of the extensions of non-neighboring sides of a **pentagon** HIJKL. Prove that the neighboring pairs of the circumcircles of the triangles ALH, BHI, CIJ, DJK, EKL intersect at 5 concyclic points P, Q, R, S, T.

Pentagon is any five-sided polygon or 5-gon. The sum of the internal angles in a simple pentagon is 540° .

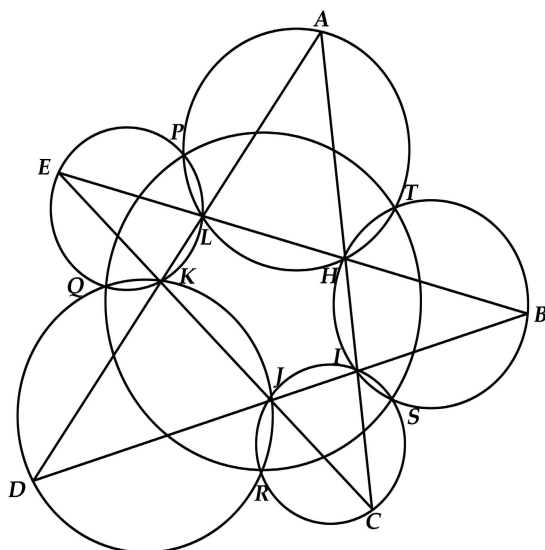
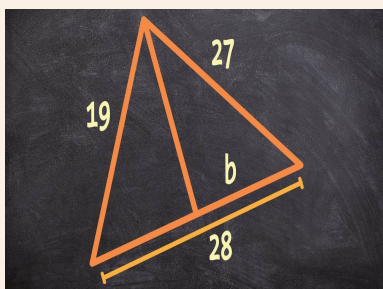


Figure 6: Miquel's 5-circle theorem.



CORE CONCEPT: PEDAL TRIANGLE

A pedal triangle is obtained by projecting a point onto the sides of a triangle. Consider a triangle ABC and a point, P , that is not one of the vertices. Then you drop perpendiculars from P to the three sides of the triangle. Label the intersections of the lines from P as L, M, N with the sides BC, AC, AB .

L, M, N with the sides BC, AC, AB .

The pedal triangle of a triangle ABC and point P is the triangle whose vertices are the projections of a P to the sides of the triangle. For any point P on the plane of a triangle ABC , the foot of the perpendiculars from P onto the sides of the triangle ABC form a triangle $A_1B_1C_1$ called the pedal triangle of the point P with respect to the triangle ABC .

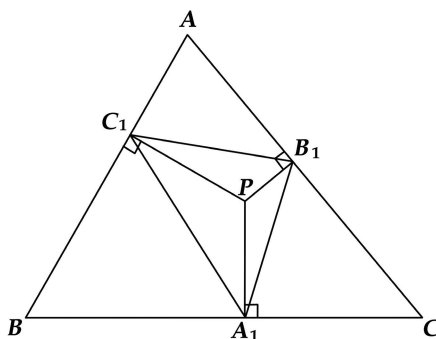
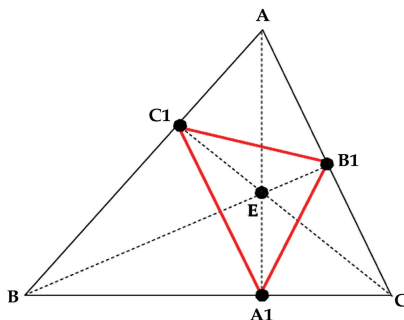


Figure 7: Pedal triangle.

Proofs for Pedal Triangles

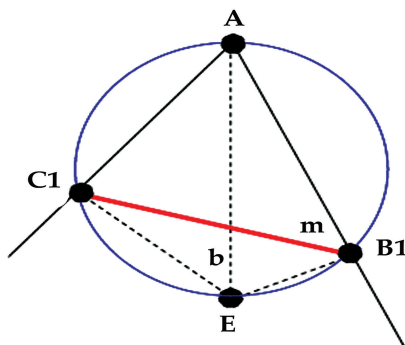
In the following diagram, the points A_1 , B_1 , and C_1 are where the altitudes from each **vertex** of the acute triangle $\triangle ABC$ intersect the opposite sides. Each of these altitudes makes an angle of 90° with the side of the triangle.



Vertex is a point where two or more curves, lines, or edges meet. As a consequence of this definition, the point where two lines meet to form an angle and the corners of polygons and polyhedra are vertices.

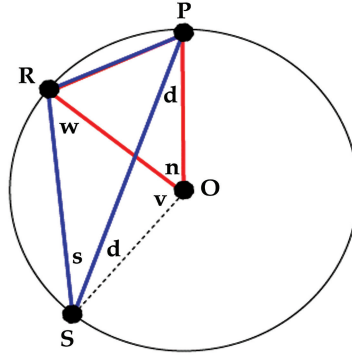
We want to show that triangles $\triangle AB_1C_1$, $\triangle A_1BC_1$, and $\triangle A_1B_1C$ are similar to triangle $\triangle ABC$. We will do this for triangle $\triangle AB_1C_1$. The argument for the other two triangles are similar. It is only necessary to show that triangles $\triangle ABC$ and $\triangle AB_1C_1$ have the same angles. Since the vertex at A is common to both triangles, that angle is the same for both triangles. We now show that $\angle B = \angle AB_1C_1$.

First, notice that $\angle AEC_1 = \angle B$ because $\triangle ABA_1$ and $\triangle AC_1E$ are both right triangles that share the angle $\angle C_1AE$.



Let $b = \angle AEC_1$ and $m = \angle AB_1C_1$. We want to show that $b = m$. In the figure above, the three points A , C_1 , and E lie on a circle. Because $\triangle AC_1E$ is a right triangle, the segment AE is a diagonal of this circle. Now the three points A , B_1 , and E also lie on a circle. Since $\triangle AB_1E$ is also a right triangle, the segment AE is a diagonal of this circle. Thus both circles must be the same since they have the same diagonal. Thus all four points actually lie on the *same* circle as shown in the figure. *Lemma:* Let points P , R , and S lie on a circle with center O . In the figure

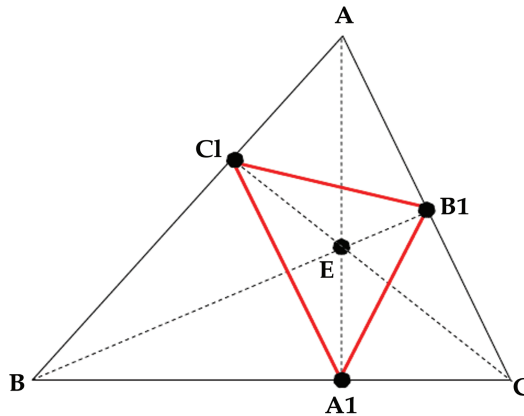
below, if the chords SR and SP form the same arc as the chords OR and OP, then the central angle n is twice the inscribed angle s .



To see why this is true, first notice that triangles $\triangle ORS$ and $\triangle OPR$ are isosceles triangles. Also observe that $2d + n + v = 180^\circ$ and $2w + v = 180^\circ$ (since $s + d = w$). Subtracting these two equations shows that $2(d - w) + n = 0$, or $2(w - d) = n$. But $w - d = s$, so we get $2s = n$.

Now apply the lemma to the situation above. Since both sets of chords from points E and B_1 form the same arc of the circle AC_1 , the lemma implies that both angles are half the central angle with that arc, and hence $b = m$.

Scaling: Consider again the picture of the pedal triangle. We want to know the scaling factor relative to the original triangle for each of the similar triangles left when the pedal triangle is removed.



Since $\triangle ABC$ is similar to $\triangle A_1B_1C_1$, the corresponding sides are proportional, i.e.

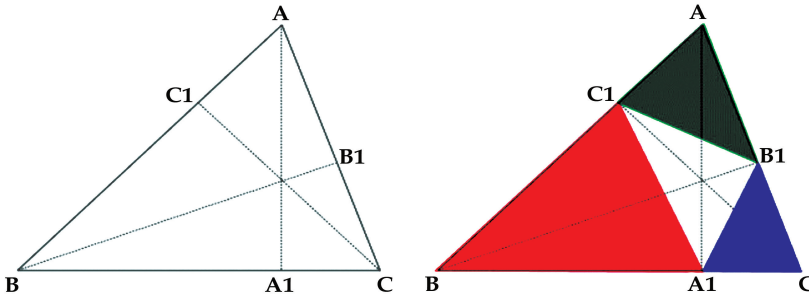
$$\frac{AB_1}{AB} = \frac{AC_1}{AC} = \frac{B_1C_1}{BC}$$

But $\triangle ABB_1$ is a right triangle, so $\cos(A) = AB_1/AB$. Thus the scaling factor for the top triangle is $\cos(A)$. The same argument shows that the scaling factor for the lower left triangle is $\cos(B)$ and the scaling factor for the lower right triangle is $\cos(C)$.

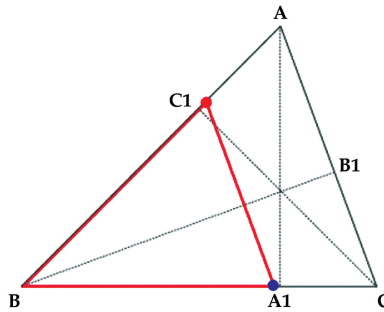
Reflections, Rotations and Translations: The IFS for the Sierpinski pedal triangle involves scaling, reflections, rotations, and translations. We can take the original tri-

angle so that vertex B is at the origin and vertex C is at the point (1,0). The triangle is then completely specified by the values of the three angles A, B, and C.

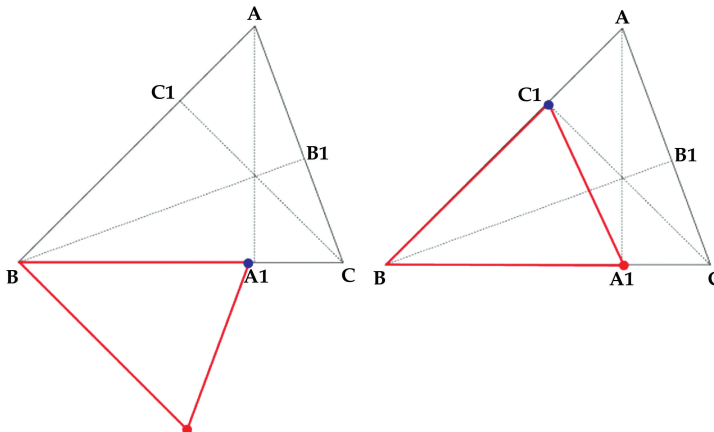
Consider the following example.



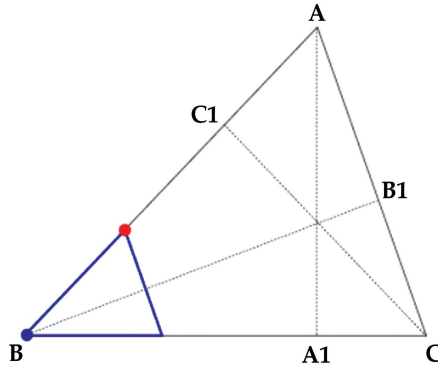
The scaling factor, as shown above, for the red triangle is $\cos(B)$. The first step is to scale the original triangle by $\cos(B)$.



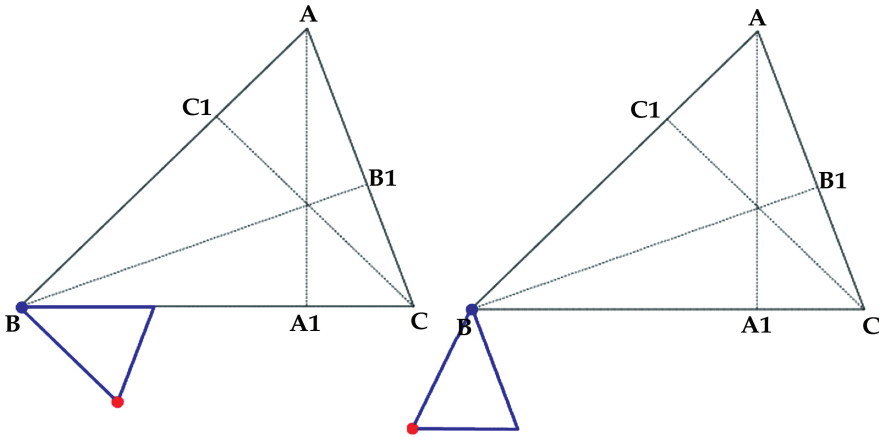
Triangle $\triangle ABC$ is similar to triangle $\triangle A_1BC_1$ as shown in the first section above. Thus the red point must go to point A_1 and the blue point must go to C_1 . To do this, first do a vertical reflection across the x-axis followed by a counterclockwise rotation through the angle B. No additional translation is needed.



The scaling factor for the blue triangle is $\cos(C)$. Scale the original triangle by $\cos(C)$.



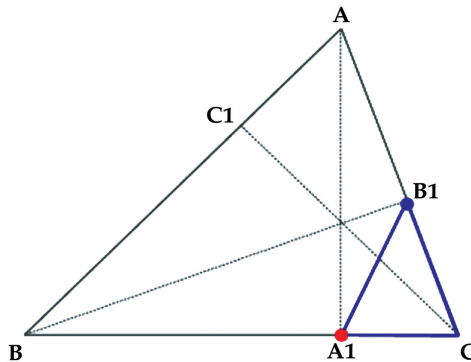
Triangle $\triangle ABC$ is similar to triangle $\triangle A_1B_1C$. Thus the red point must go to A_1 and the blue point must go to B_1 . To do this, first do a vertical reflection across the x-axis followed by a clockwise rotation through the angle C .



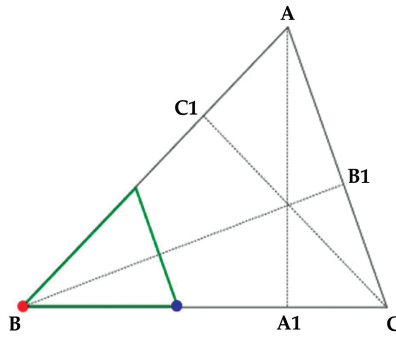
The final step is to translate the triangle so that the blue point goes to B_1 . To find the coordinates (x,y) of B_1 , first notice that $B_1C=BC \cdot \cos(C)=\cos(C)$ and therefore

$$x = 1 - B_1C \cdot \cos(C) = 1 - \cos^2(C) = \sin^2(C)$$

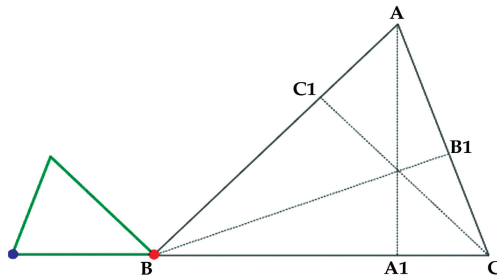
$$y = B_1C \cdot \sin(C) = \sin(C) \cos(C)$$



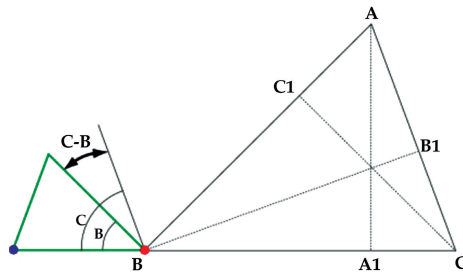
The scaling factor for the green triangle is $\cos(A)$. Scale the original triangle by $\cos(A)$.



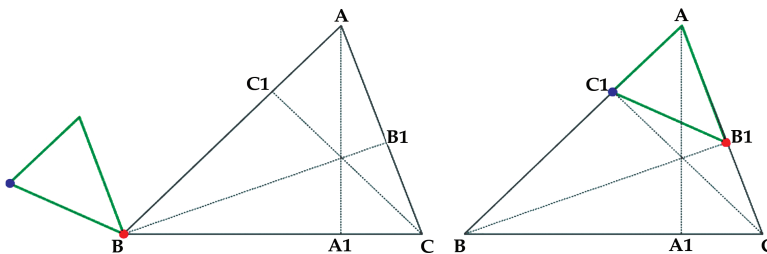
Triangle $\triangle ABC$ is similar to triangle $\triangle AB_1C_1$. Thus the red point must go to B_1 and the blue point must go to C_1 . To do this, first do a horizontal reflection across the y -axis.



Now we need to rotate the green triangle so that the side from the top vertex to the origin becomes parallel to side AC . As the following figure shows, this involves rotating the green triangle clockwise through the angle $C-B$. (If $C < B$, then the rotation would be counterclockwise through the angle $B-C$.)



The final step is to translate the green triangle so that the red point goes to point B_1 . This translation is the same as for the blue triangle.



Theorem

Let $A_1B_1C_1$ be the pedal triangle of the point P with respect to the triangle ABC . Then

$$(A_1B_1C_1) = \frac{R^2 - OP^2}{4R^2} (ABC),$$

where O is the circumcentre and R is the circumradius of the triangle ABC

Proof: Extend BP meeting the circumcircle of $\triangle ABC$ at B_2 . Join B_2C . As in the figure, $\angle A_1 = \alpha + \beta = \angle B_2CP$.

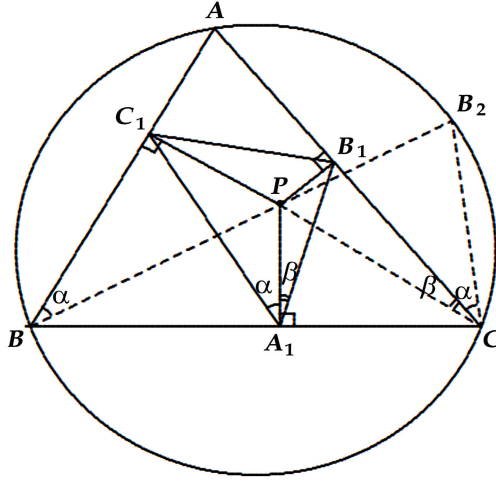


Figure 8: Area of the pedal triangle.

Thus

$$(A_1B_1C_1) = \frac{1}{2} A_1B_1 \cdot A_1C_1 \cdot \sin A_1 = \frac{1}{2} (PC \sin C)(PB \sin B) \sin \angle B_2CP$$

Also

$$\frac{\sin \angle B_2CP}{\sin A} = \frac{\sin \angle B_2CP}{\sin \angle BB_2C} = \frac{PB_2}{PC}$$

Thus

$$\begin{aligned} (A_1B_1C_1) &= \frac{1}{2} PB_2 \cdot PB \sin A \sin B \sin C \\ &= \frac{1}{2} (R^2 - OP^2) \sin A \sin B \sin C \\ &= \frac{R^2 - OP^2}{4R^2} (ABC). \end{aligned}$$

The above result is a generalization of Simson's theorem.

Corollary: The point P lies on the circumcircle of $\triangle ABC$ if and only if the area of the pedal triangle is zero if and only if A_1, B_1, C_1 are collinear.

Exercise 7

Show that the third pedal triangle is similar to the original triangle.

Exercise 8

Let P be a point on the circumcircle of the triangle ABC . Prove that its Simson line with respect to the triangle ABC bisects PH , where H is the orthocentre of the triangle ABC .

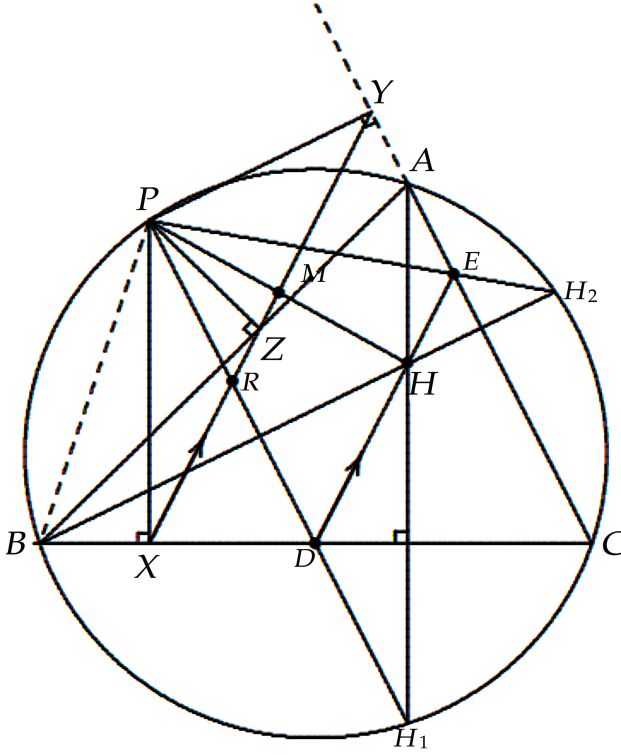


Figure 9: The Simson line bisects PH .

Exercise 9

Let P and P' be diametrically opposite points on the circumcircle of the triangle ABC . Prove that the **Simson lines** of P and P' meet at right angle on the nine-point circle of the triangle

Exercise 10

Prove Brahmagupta-Mahavira formula: Let $ABCD$ be a cyclic quadrilateral with $AB = b$, $BC = c$, $CD = d$, $DA = a$ and $AC = m$, $BD = n$. Then

$$\frac{m}{n} = \frac{ab + cd}{ad + bc}$$

Simson lines are the line containing the feet, and of the perpendiculars from an arbitrary point on the circumcircle of a triangle to the sides or their extensions of the triangle.

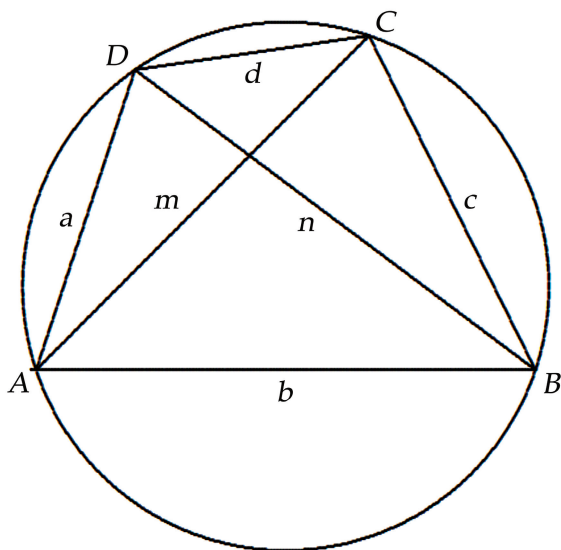
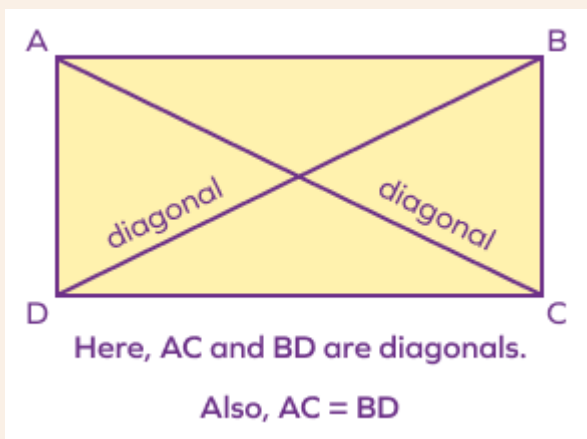


Figure 10: Brahmagupta-Mahavira formula.

CONCEPT OF DIAGONAL OF A RECTANGLE

The line segments that join the opposite corners of a rectangle are called diagonals. In the given figure, the two diagonals of the rectangle are AC and BD. The diagonals of a rectangle are the same in length. Therefore, $AC = BD$.



If we know the length and width of a rectangle, we can use Pythagorean Theorem to find the length of its diagonal.

In the given figure, ADB forms a triangle right-angled at A. The diagonal (BD) of the rectangle forms its hypotenuse.

so, using the pythagorean theorem, we get,

$$\text{diagonal}^2 = \text{length}^2 + \text{width}^2$$

$$\therefore \text{diagonal} = \sqrt{\text{length}^2 + \text{width}^2}$$

SUMMARY

- A quadrilateral is a closed geometric figure with four sides and four vertices. The segments of a quadrilateral intersect only at their endpoints. Special types of quadrilaterals include squares and rectangles.
- A concave quadrilateral contains a reflex angle (an angle greater than 180°), whereas all of the angles in a convex quadrilateral are less than 180° .
- A parallelogram is a quadrilateral with opposite sides parallel. But there are various tests that can be applied to see if something is a parallelogram. If a quadrilateral has two pairs of parallel opposite sides, then it is called a parallelogram.
- A rectangle is a quadrilateral with four right angles. It can also be defined as an equiangular quadrilateral, since equiangular means that all of its angles are equal ($360^\circ/4 = 90^\circ$). It can also be defined as a parallelogram containing a right angle. A rectangle with four sides of equal length is a square. The term oblong is occasionally used to refer to a non-square rectangle.
- A Square is a flat shape with 4 equal sides and every angle is a right angle (90°). A square is a regular quadrilateral, which means that it has four equal sides and four equal angles (90-degree angles, or (100-gradian angles or right angles).
- A rhombus is a simple quadrilateral whose four sides all have the same length. Another name is equilateral quadrilateral, since equilateral means that all of its sides are equal in length.
- A trapezoid is a 4-sided flat shape with straight sides that has a pair of opposite sides parallel. A trapezium is defined by the properties it does not have. It has no parallel sides. Any quadrilateral drawn at random would probably be a trapezium.
- A quadrilateral is called a convex quadrilateral, if the line segment joining any two vertices of the quadrilateral is in the same region.
- Ptolemy's Theorem states that, in a cyclic quadrilateral, the product of the diagonals is equal to the sum the products of the opposite sides.

MULTIPLE CHOICE QUESTIONS

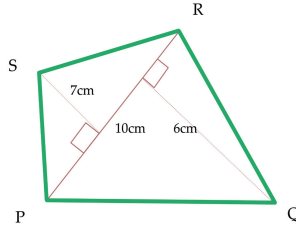
1. **The quadrilateral whose all its sides are equal and angles are equal to 90 degrees, it is called:**
 - a. Rectangle
 - b. Square
 - c. Kite
 - d. Parallelogram
2. **The sum of all the angles of a quadrilateral is equal to:**
 - a. 180°
 - b. 270°
 - c. 360°
 - d. 90°
3. **A trapezium has:**
 - a. One pair of opposite sides parallel
 - b. Two pairs of opposite sides parallel to each other
 - c. All its sides are equal
 - d. All angles are equal
4. **A rhombus can be a:**
 - a. Parallelogram
 - b. Trapezium
 - c. Kite
 - d. Square
5. **A diagonal of a parallelogram divides it into two congruent:**
 - a. Square
 - b. Parallelogram
 - c. Triangles
 - d. Rectangle

Answer

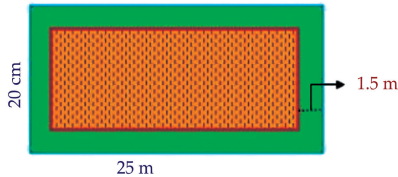
1. (b) 2. (c) 3. (a) 4. (d) 5. (c)

REVIEW QUESTIONS

1. Discuss about quadrilaterals and its type.
2. Define the convex and concave quadrilaterals
3. Explain the Ptolemy's theorem and with example.
4. How are rhombuses and squares different?
5. Find the area of a quadrilateral PQRS, whose diagonal and the perpendicular distances are as shown given.



6. Find the area of a quadrilateral one of whose diagonal is 18 cm and the perpendiculars from the other vertices to the given diagonal are 7 cm and 5 cm respectively.
7. Find the area of a parallelogram one of whose sides is 35 cm and the corresponding height is 8 cm.
8. A rectangular plot 25 m long and 20 m wide is to be covered with grass leaving 1.5 m all around it. Find the area of the grass lawn.

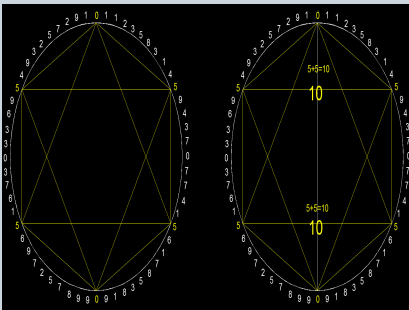


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CHAPTER 3

INVERSIVE GEOMETRY



Inversive geometry is the study of those properties of figures that are preserved by a generalization of a type of transformation of the Euclidean plane, called inversion. These transformations preserve angles and map generalized circles into generalized circles, where a generalized circle means either a circle or a line (loosely speaking, a circle with infinite radius). Many difficult problems in geometry become much more tractable when an inversion is applied.

After studying this chapter, you will understand the core concepts of:

- Orthogonality
- Inversion
- Concentric Circles

INTRODUCTION

Inversive geometry is a field of study centered around the inversion - inversive transformation. The notion of circle or sphere can also be equivalently expressed as the 4-ary relation of circularity, (the relation between 4 points saying they belong to the same circle or straight line) suffices to define angles of intersection, for the following intuitive reason:

In n -dimensional geometry, let us start from the concept of $(n-1)$ -dimensional sphere, which are circles if $n=2$. A space with such a concept is smooth (as small parts of non-small spheres or circles are approximately straight) and thus can compare nearby small volumes (areas). Then, to measure the angle of intersection between 2 spheres (circles) S and S' at a given point, consider a small ball (disk) around this intersection, divided by S and S' into 4 pieces, such that each has the same volume (area) as the opposite piece. Then the ratio of the volume of one piece to that of the whole ball, multiplied by 2π , approaches the angle in radians of that piece, when the volume of the whole ball approaches zero.

In an affine plane, measures of angles suffice to define the concept of circle centered on a point O , as a curve C equivalently satisfying:

$\forall A, B \in C$, the angle between the tangent lines to C at A and B , equals (with the same sign) the angle between straight lines (OA) and (OB) .

For a variable $B \in C$ (while $A \in C$ may vary or not), the angle between straight lines (OA) and (OB) is proportional to the area of the region limited by $[OA]$, $[OB]$ and the part of C between A and B .

Now you may ask, how can we have a concept of circles but no affine structure? The alignment relation may fail to be defined by “to not be contained in a same circle” if straight lines are confused among circles: straight lines are the limit cases of circles that “open” up to infinity. For any two circles (or one straight line and one circle) that intersect at one point, either it is a point of tangency (the angle of intersection is zero), or they intersect at one other point with the same angle. Thus the confusion of straight lines among circles requires that any two intersecting straight lines also intersect at another point, that is located “at infinity” (think of a fixed line intersecting a moving circle at a fixed point with a fixed angle, and let that circle open up to become a straight line, and again a circle on the other side: the other intersection goes through infinity).

A typical automorphism of this geometry, is the inversion, defined by

$$(x,y) \mapsto (x/(x^2+y^2), y/(x^2+y^2)).$$

It is called this way because the image of every point is a point in the same direction from the chosen center, and with a distance from the center given by the inverse of the original distance. (The inversion reverses the orientation; we can get an automorphism preserving the orientation, by composing it with a reflection). It can be similarly defined in Euclidean spaces of any dimension, with the same properties.

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \quad \text{Determine if matrix } A \text{ is orthogonal.}$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \quad A^T = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

INVERSE

CORE CONCEPT: ORTHOGONALITY

Orthogonality a property synonymous with perpendicularity when applied to vectors but applicable more generally to functions. Two elements of an inner product space are orthogonal when their inner product—for vectors, the dot product; for functions, the definite integral of their product—is zero. A set of orthogonal vectors or functions can serve as the basis of an inner product space, meaning that any element of the space can be formed from a linear combination of the elements of such a set.

We shall study circles which intersect at right angles. As we shall see, such circles gives a generalization of the definition of inversion. The concept of **orthogonal circles** plays an important role in the theory of inversion.

The two supplementary angles between two circles are naturally defined as the angles between their tangents at a point of intersection. If two circles intersect at P and Q, then the angles at P and Q are easily seen to be equal by the reflection in the line of the centers.

Orthogonal circles are orthogonal curves, i.e., they cut one another at right angles. By the Pythagorean theorem, two circles of radii and whose centers are a distance apart are orthogonal if.

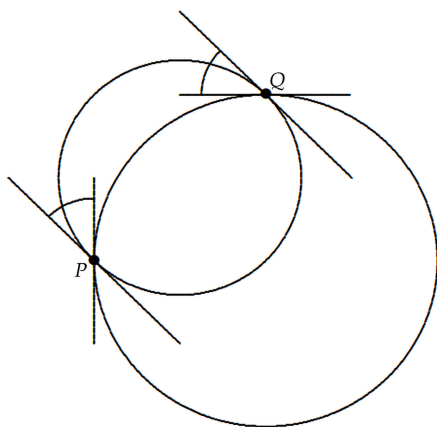


Figure 1: The angles between two intersecting circles.

To see how angles are affected by inversion in a circle ω with center O, let θ be one of the angles between two lines a and b through the point P. See Figure 2. The line a is inverted to a circle α passing through O whose tangent at O is parallel to a. Similarly, the line b is inverted to a circle β passing through O whose tangent at O is parallel to b. Since θ is one of the angles between these tangents at O, it is one of the angles of

intersection of α and β . As α and β also intersect P' , the inverse of P , the same angle θ also appears at P' .

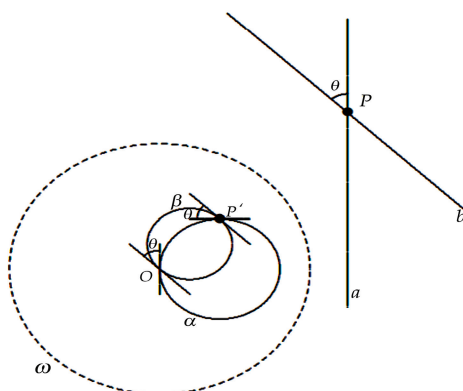


Figure 2: Inversion preserves angles.

Theorem

Any point on ω is its own inverse; the inverse of any other point P is the second intersection of any two circles through P orthogonal to ω .

Proof. Let P be a point not on ω , and let α and β be two circles through P orthogonal to ω .

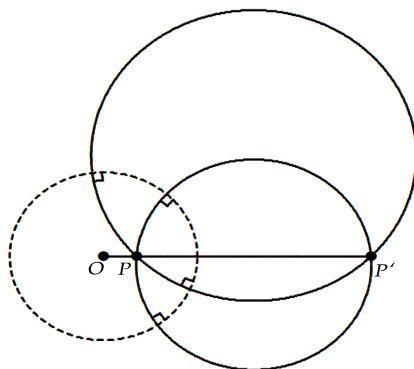
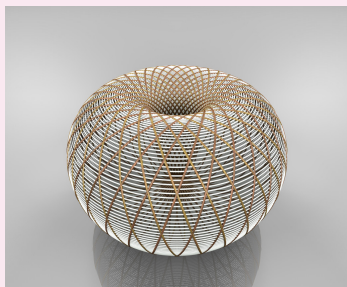


Figure 3: Inversion by means of orthogonal circles.

As α inverts back to itself, P' lies on α . Similarly P lies on β . Thus P' is the second point of intersection between α and β . (Thus if O is the center of ω , then O , P , P' are collinear and $OP \cdot OP' = k^2$, where $k = \text{radius of } \omega$.)



CORE CONCEPT: INVERSION

Inversion is the process of transforming points P to a corresponding set of points P' known as their inverse points. Two points P and P' are said to be inverses with respect to an inversion circle having inversion center $O = (x_0, y_0)$ and inversion radius k if P' is the perpendicular foot of the altitude of $\triangle OQP$, where Q is a point on the circle such that $OQ \perp PQ$.

Given a circle ω with center O and radius k and a point P different from O , we define the “inverse” of P to be the point P' on the ray OP such that $OP \cdot OP' = k^2$.

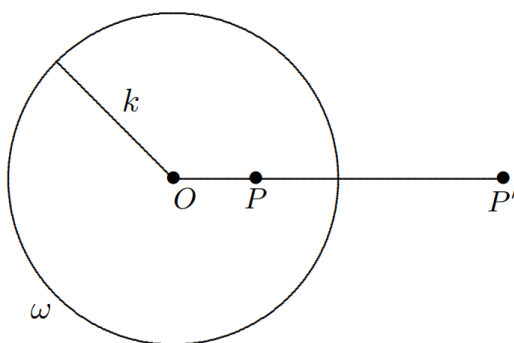


Figure 4: Inversion.

Basic Properties

1. Inversion is a map from $\mathbb{R}^2 \setminus \{O\}$ onto itself.
2. $(P')' = P$. Thus inversion is a bijection of order 2
3. If P is inside (outside) ω , then P' is outside (inside) ω
4. If P lies on ω , then $P' = P$. That is P is fixed under the inversion with respect to ω
5. If α is a circle centered at O with radius r , then α' is also a circle centered at O with radius k^2/r .
6. Any line ℓ through O is its own inverse. That is $\ell' = \ell$.
7. Let $P \neq O$ be a point inside ω . To determine the position of P' on the ray OP , draw a chord through P perpendicular to OP meeting ω at T and S . Then the tangents to ω at T and S meet at the inverse point P' .

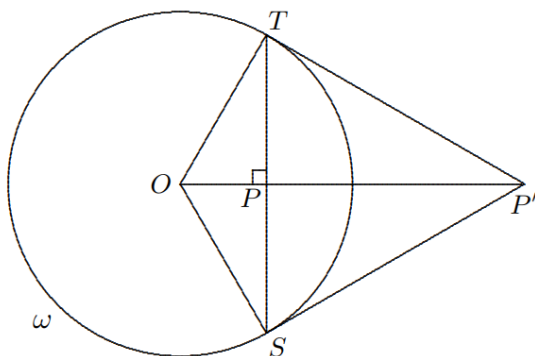


Figure 5: The inverse of a point.

8. If P is a point outside ω , then the two tangents from P to ω determine the chord TS . The intersection between TS and the ray OP gives the inverse point P' . Alternatively, draw the circle with diameter OP intersecting ω at T and S . Then P' is the midpoint of TS .
9. The inverse of any line ℓ , not through O , is a circle through O (minus the point O itself), and the diameter through O of this circle is perpendicular to ℓ .

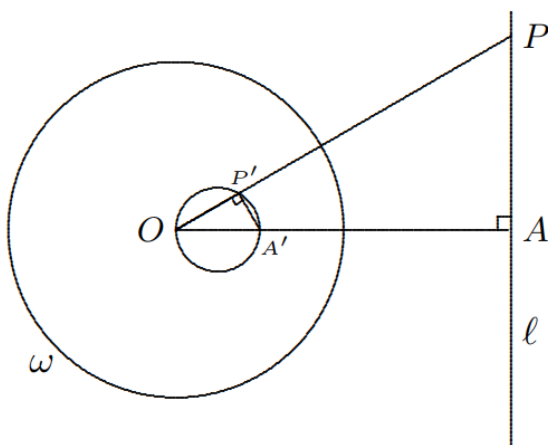


Figure 6: The inverse of a circle.

10. The inverse of any circle through O (with O omitted) is a line perpendicular to the diameter through O . That is a line parallel to the tangent at O to the circle.
11. A pair of intersecting circles α and β , with common points O and P inverts into a pair of intersecting lines α' and β' through the inverse point P' .

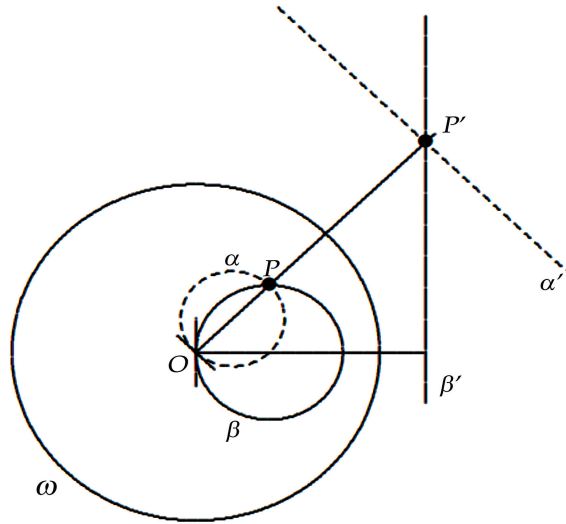


Figure 7: The inverses of a pair of circles intersecting at the center of inversion.

12. If α and β are tangent to each other at O , then α' and β' are parallel.

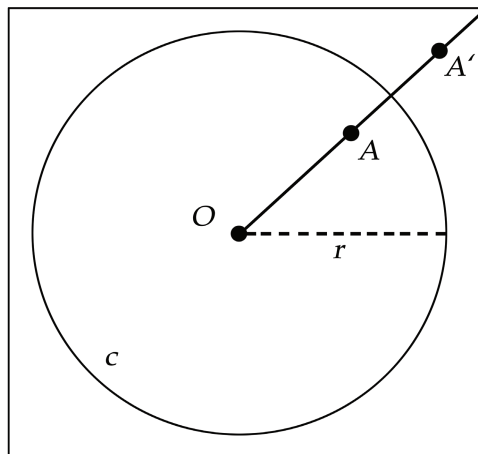
Circle Inversion

If the inversion circle c has radius r and center O , then the inverted point A' of a point A lies on a ray from O and A , and

$$OA \cdot OA' = r^2.$$

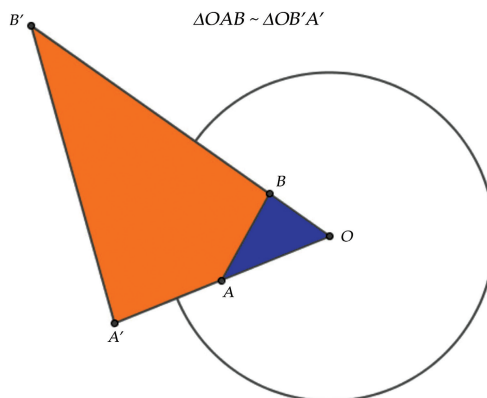
As a point A gets closer and closer to O , the inverted point A' tends to infinity. The inversion of O is not defined unless a point at infinity is introduced. Points on the circle c are inverted to themselves. Points inside c are inverted to points on the outside, and vice versa. Inversion in a circle is a transformation that flips the circle inside out.

It is possible to construct the inverted point using a ruler and compass. In GeoGebra you can use the tool Reflect Object in Circle to create the inverted point.



Similar Triangles

Let O be the center of the inversion circle. Let A and B be distinct points not equal to O . Consider the triangles $\triangle OAB$ and $\triangle OB'A'$.

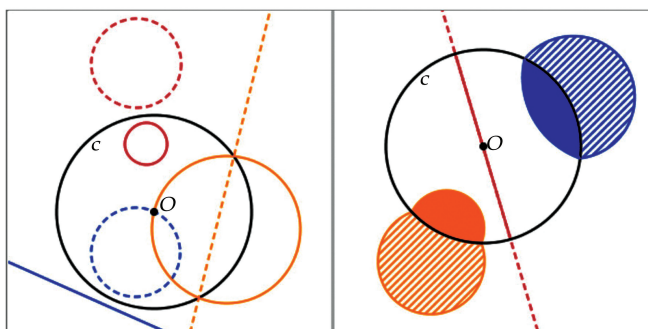


We know that $OA \cdot OA' = OB \cdot OB' = r^2$. Since $\frac{OA}{OB} = \frac{OA'}{OB'}$ and $\angle AOB = \angle A'OB'$ the SAS-criterion for similar triangles holds. Hence, the triangles are similar.

Generalized Circles and Angle Preservation

Circle is a simple closed shape. It is the set of all points in a plane that are at a given distance from a given point, the Centre; equivalently it is the curve traced out by a point that moves so that its distance from a given point is constant.

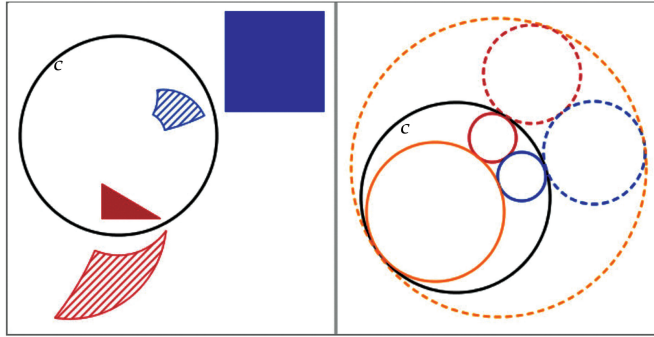
Let O be the center of the inversion circle. A circle through O is inverted to a line. Conversely a line is inverted to a circle through O . A circle that does not contain O is inverted to a **circle**. It is convenient not to distinguish between circles and lines, and instead use so called generalized circles. GeoGebra can handle generalized circles. Try it out by using the tool Reflect Object in Circle on circles and lines!



Circle and lines are inverted to circles or lines.

Points on the inside of c are inverted to points on the outside.

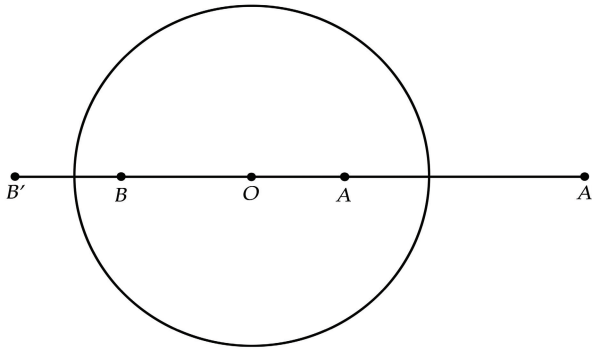
Angles are preserved when points are inverted in a circle. As a consequence, objects that are tangent to each other, are inverted to objects that are also tangent to each other.



Angles are preserved and so is tangency.

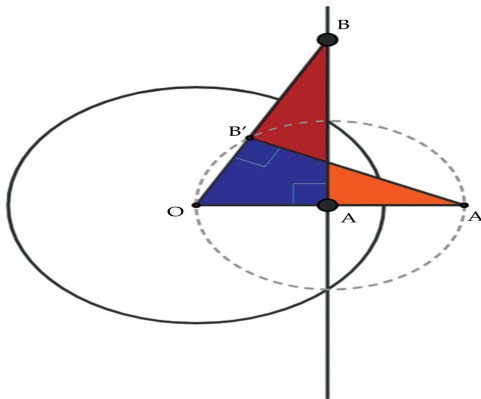
Lines and Circles Through the center of the Inversion Circle

If two points A and B are both on a line l through the center O ; then both rays from O to A , and from O to B are on l . Hence the inverted points A' and B' are also both on l . Lines through the center O are mapped onto themselves when inverted in a circle.



Lines through O are inverted to lines through O .

Let l be a line not through O . Make another line through O perpendicular to l and let A be the intersection between the lines. Let B be a point on l .



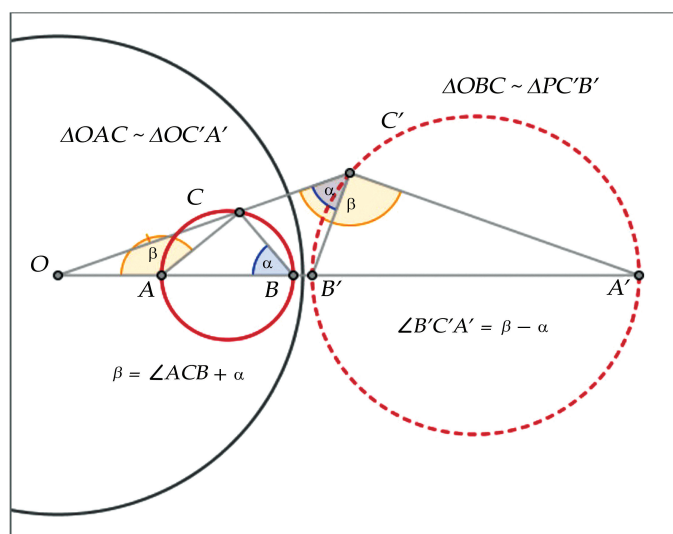
Lines not through O are inverted to circles through O , and vice versa.

Move B along the line. Move the point A.

Since $\triangle OAB \sim \triangle OB'A'$, we must have that $\angle OB'A' = \angle OAB = 90^\circ$. Using Thales theorem and the converse of Thales theorem, we see that B' is on a circle through A and A' . We have shown that lines not through O are inverted to circles through O (excluding the point O). By reversing the process, we can show that circles through O are inverted to lines not through O.

Circles that are not Through the Centre of the Inversion Circle

Let c be a circle not through O. Let A be the point on c closest to O and let B be the point on c furthest away from O. Let C be any point on c distinct from A and B. The figure below shows the case when c is inside the circle of inversion.



Thales's theorem states that if A, B, and C are distinct points on a circle where the line AC is a diameter, then the angle $\angle ABC$ is a right angle.

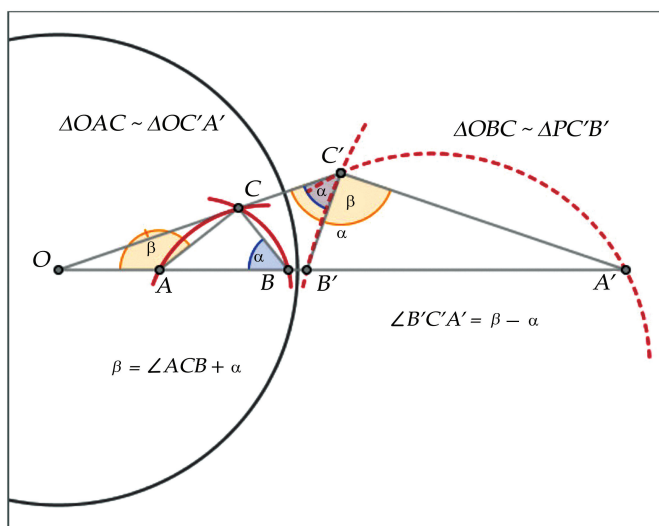
$$\angle ACB = 90^\circ \implies \angle B'C'A' = 90^\circ.$$

Using similarity, Thales theorem, and the converse of **Thale's theorem**; it is possible to show that C' is on a circle through A' and B' . The proof is sketched in the figure above but the details are omitted. Other placements of c can be sketched in an analogous way. In summary: circles not through O are inverted to circles.

Angles are Preserved

The angle between two circles is the angle between the tangent lines of the circles at an intersection point. Instead of using the angle between tangent lines, we will use the

angle $\angle ACB$ where C is an intersection point, and the points A and B are on a line and also on one of the circles; as in the figure below.



$$\angle ACB = \angle B'C'A'.$$

The angle between the tangent lines of the circles, is the limit of the angle $\angle ACB$ as the line through A and B gets closer and closer to the line through O and C . It is possible to show that $\angle ACB = \angle B'C'A'$. The proof is sketched in the figure above but the details are omitted. Other cases of angles between circles or lines can be proved in an analogous way. In summary, the angle between two generalized circles is preserved when the generalized circles are inverted in a circle.

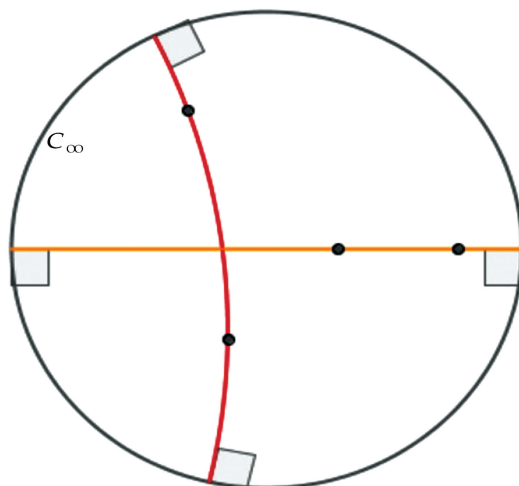
Circle Inversion when the Circles are Perpendicular to Each Other

The **Poincaré disc** (in 2D) is the area inside a circle. The circle is not included. When using the Poincaré disc model, only points in the Poincaré disc are considered. The Poincaré disc makes up the entire world. The boundary of the disc is the circle at infinity, C_∞ .

A geodesic through points A and B is defined as the circular arc through A and B that is perpendicular to C_∞ . If A and B lie on the diameter of C_∞ , that diameter is the geodesic through A and B . A geodesic is the hyperbolic equivalence of a Euclidean line. A geodesic is a hyperbolic line.

Poincaré disc

model, also called the conformal disk model, is a model of 2-dimensional hyperbolic geometry in which all points are inside the unit disk, and straight lines are either circular arcs contained within the disk that are orthogonal to the unit circle or diameters of the unit circle.



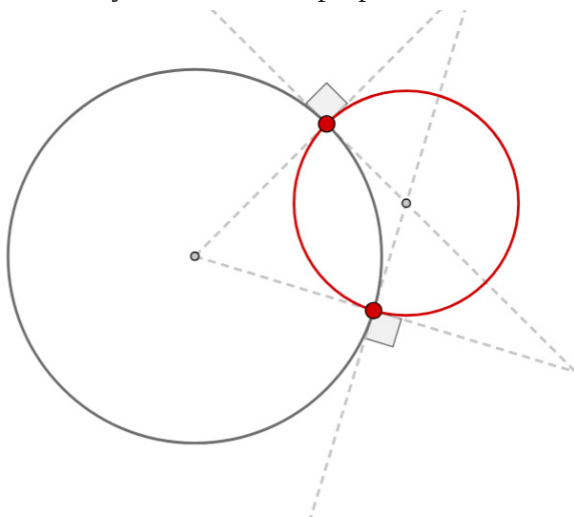
Geodesics through two points.

The reason why perpendicular generalized arcs are used as hyperbolic lines to model hyperbolic geometry, is because of the properties of perpendicular circles when doing circle inversion.

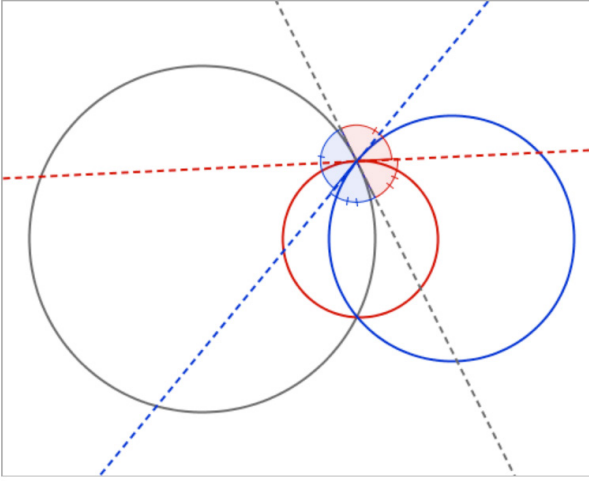
Circles Perpendicular to the Circle of Inversion are Inverted to themselves

Given a circle c and two distinct points on c ; then there is a unique circle through the points that is perpendicular to c . The proof is omitted but sketched in the worksheet below.

If the red circle is inverted in the gray circle, the big red points are inverted to themselves, and the inverted circle is also perpendicular to the gray circle (since angles are preserved). Since there is only one unique circle through the points, perpendicular to the gray circle, the inverted circle must be the same as the original circle. A circle is inverted to itself, if and only if the circle is perpendicular to the circle of inversion.



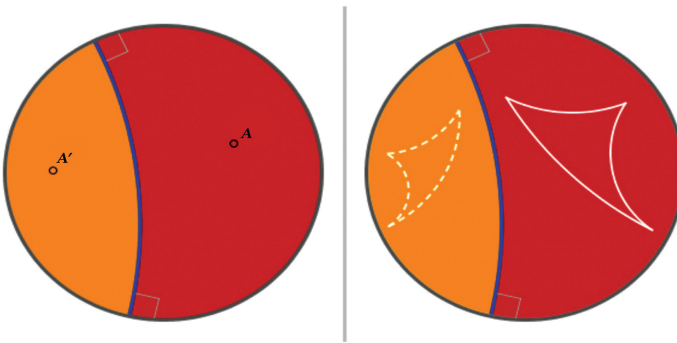
For a circle not perpendicular to the inversion circle, the inverted circle will still have the same angles to the inversion circle. Since there are two angles between two lines, and since the orientation of points is changed when inverting in a circle; the inverted circle will not be the same as the original circle. A circle is inverted to itself, if and only if the circle is perpendicular to the circle of inversion.



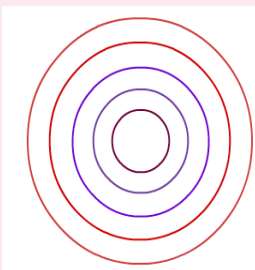
The red circle is inverted to the blue circle.

Consider a geodesic on the Poincaré disc. The geodesic is perpendicular to C_∞ , so C_∞ is inverted to itself when inverted in the geodesic. If a point inside C_∞ is inverted in a geodesic, the inverted point will also be on the inside of C_∞ , on the other side of the **geodesic**.

A **geodesic** is a locally length-minimizing curve. Equivalently, it is a path that a particle which is not accelerating would follow. In the plane, the geodesics are straight lines.



The red area is inverted to the yellow area. Angles are preserved.

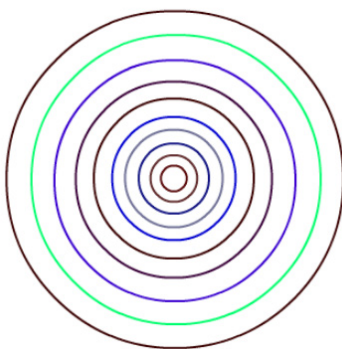


CORE CONCEPT: CONCENTRIC CIRCLES

Concentric circles are circles with a common center. The region between two concentric circles of different radii is called an annulus. Any two circles can be made concentric by inversion by picking the inversion center as one of the limiting points.

A circle is a set of all points that are at an equal distance from a fixed point in a given plane. The fixed point is termed as the center of the circle and the common equal distance is known as the radius, the plural of which is radii. Concentric circles are those circles that have a common center for all its circles. Concentric circles are circles of different sizes. But all these circles have a common center. A concentric circle does not mean that the entire circles are of same radius but yes they are having same center. If all circles have same radius then they will not be concentric they will be lying on one another like a single circle.

When two or more circles have the same center but have different radii, they are called as concentric circles, that is, circles with common center.



Two circles can easily be made concentric by the process of inversion that is, by picking the center of inversion as one of the points of limiting.

Examples of Concentric Circles:

The circles that appear on a dartboard or on an archery target or of other similar sports are all concentric. Also, the ripples that are formed on the surface of the water whenever a stone is dropped on the surface of the water are also concentric.

Properties of Concentric Circles

Circles having the same center but different radii are called concentric circles.

Properties of Concentric Circles:

- 1) Concentric circles always fit inside each other, the one with smaller radius in the one with bigger radius starting from the biggest circle to the smallest one.
- 2) Concentric circles are always equidistant from each other throughout the way around. The distance can be calculated by subtracting the radii the radius of inner circle from the radius of the outer circle
- 3) **Circular Ring:** The region lying between two concentric circles is called annulus or the circular ring. The area of the ring can be calculated easily when the radii of both the circles is given. For that we subtract the area of the inner circle from the area of the outer circle.

Let 'R' is the radius of the outer circle and 'r' is the radius of the inner circle. That is:

$$\text{Area of ring, } A = \pi (R^2 - r^2)$$

π (pi) is a constant with value 3.14 approximately.

4) Tangent to the Inner Circle:

The tangent drawn at a point to the inner circle always becomes a chord to the outer circle. The point of contact to the inner circle is the point of tangency which becomes the middle point of the chord to the outer circle.

Solved examples:

Example 1: Find the area of the circular ring formed by two concentric circles, given that the radii of the outer and the inner circles are 4 cm and 2 cm respectively.

Solution: The area of the region enclosed by two concentric circles is given by:

$$A = \pi (R^2 - r^2)$$

Here, $R = 4$ cm, $r = 2$ cm and $\pi = 3.14$

So Area, $A = 3.14 (4^2 - 2^2)$ sq cm = $3.14 (16 - 4)$ sq cm = 37.68 sq. cm.

Example 2: Find the length of the chord in the outer circle which is the tangent to the inner circle when radius of the inner circle is 6 cm and the radius of the outer circle is 10 cm.

Solution:

Here, radius of inner circle $OT = 6$

Radius of outer circle $OB = 10$.

We need to find AB.

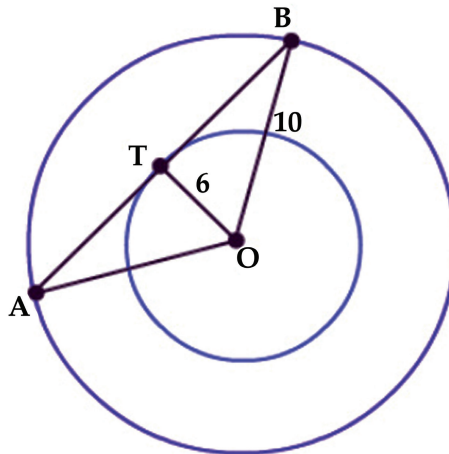
Since the point of tangency is the mid-point of the chord also. Thus T bisects AB.

Hence, $AT = TB$.

Also we know that the radius of a circle is perpendicular to the tangent at the point of contact, thus, angle $OTB = 90^\circ$.

This implies, triangle OTB is a right triangle and so by Pythagoras theorem we get,
 $TB = 8$ cm.

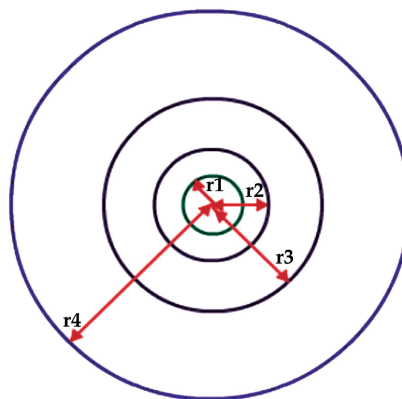
So, $AB = 2 \times 8 = 16$ cm. Hence the length of the tangent is 16 cm



Concentric Circles Formula

Concentric circles mean all those circles which have common center but different radii. It is interesting to note that there can be n number of concentric circles. The value of n can approach to infinite or we can say it has infinite value. Below illustration shows four concentric circles with same center and different radii r_1 , r_2 , r_3 and r_4

Concentric circles:



Above figure represents the 4 concentric circles. From figure it is clear that all four circles are of different radius but all circles are co-centric. By the process of inversion, four circles can easily be drawn concentric. This is basically done by choosing center of inversion as single point of limiting.

Below are some formulas of Concentric Circles:

- 1) Concentric circles are also known as uniform concentric circles if their radii varies in a uniform way. For example from figure if $r_2/r_1 = r_3/r_2 = r_4/r_3$. So from this we can see that if ratios are equal then we can say concentric circles are uniform concentric circles in terms of the distance between any two concentric circles.

Distance between any two Concentric Circles:

Distance between any two concentric circles is expressed as the difference of their radii.

For Example: from figure if we assume C_1 as the smallest circle, C_2 as next circle to C_1 , C_3 as next to C_2 and C_4 the largest circle. Then

- 1) For circles C_1 and C_2 , the distance between these two will be equal to difference in their radii that is $r_2 - r_1$.
- 2) For circles C_1 and C_4 , the distance between these two will be equal to their radii that are $r_4 - r_1$.

Concentric Circles Area:

The concentric formula depends upon the basic area of circle formula but actually it depends again on the difference of outermost and innermost concentric circle area. We can easily see that there are four concentric circles C_1 , C_2 , C_3 and C_4 of radii r_1 , r_2 , r_3 and r_4 respectively. Here C_1 and C_4 are innermost and outer most respectively.

We know that the area of circle = $\pi \times r^2$

Here $\pi = 3.14$

So Area of complete figure = $\pi (r_4^2 - r_1^2)$

Series of Concentric Circles

Circles are said to be concentric if they have same center but they don't have same radius. Series of concentric circles can be drawn very easily because all we have to do is to use a center for the rest to draw.

There are other types of circles too like circumscribed and others. Concentric circles have the same properties like circles. But in the case of concentric circle it differs a bit because of different radius. Circles radius and circumference are proportional to each other. It is also same for concentric circles.

A series of concentric circles contains infinite number of circles. A circle which is having radius equal to one then it is called unit circle.

Area of Concentric Circles

Concentric circles are the circles which share the same center point but every circle has different radius for it. As we all know that circle is a two dimensional figure so when we draw concentric circle then we have different radius. But in case of three dimensions we don't need different radius it will look concentric.

Area of the concentric circles can be derived from the area of a circle:

Area of a circle having radius r is πr^2

The area or region which is enclosed between two concentric circles is known as annulus or circular ring. We can calculate the area of annulus by subtracting area of bigger circle from the area of smaller circle.

Concentric circles can be seen in many structures which we see in our day to day life. Like archery target it is made on the principle of concentric circle. Another example where we see the formation of concentric circle is when we drop or throw a stone in still water. The water disperse by making concentric circle.

How to Draw Concentric Circles

After having fair idea of concentric now we are in position to understand the steps or methods involved in drawing the concentric circles.

Steps/ Method to draw concentric circles:

Steps are most important to be incorporated along with the drawing of concentric circles. This is because steps provide the clear procedure or method to draw the concentric circles. Without steps it would be difficult for all users to understand the concept behind the drawing of concentric circles.

To draw concentric circles we should have clear idea of radii of the circles. First of all center point needs to be defined or located. From the center and with the help of scale we can open the compass equal to radius of the smaller circle and then circle can be drawn easily. Similarly other circles of different radii can be drawn.

Sometimes the compass has limitation of opening in that case it is preferred to use engineering set up tool box. With the help of engineering set up tool box one can achieve up to higher number of concentric circles or higher radii outermost concentric circles.

5 Concentric Circles

5 concentric circles are basically defined as the set of all points which have a single fixed point known as center of the circle. In simple words it can be defined as 5

Let r_1 and r_2 be the radius of two concentric circles. Then

Area of two concentric circles = $\pi r_2^2 - \pi r_1^2 = \pi (r_2^2 - r_1^2)$

Here π is constant value and it is approximately equal to 3.14 or $22/7$.

numbers of circles with common center and different radius of each circle.

5 concentric circles:

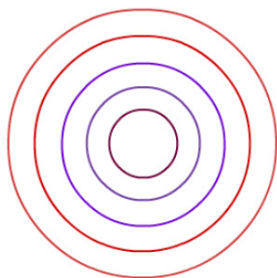


Figure here represents the 5 concentric circles. It is clear from the above figure that all 5 circles of different radius are co-centric. By the process of inversion, 5 circles can easily be drawn concentric. This is basically done by selecting or opting center of inversion as single point of limiting.

Some Basic Characteristics of 5 Concentric Circles:

- 1) In case of 5 concentric circles, all circles are placed in such a way that the smaller radius circle fits in the next adjacent higher radius of the circle. The largest radius circle is placed in such a way that it comes exactly in the outer most position. Someway it looks similar like the electronic configuration of 5 shells K, L, M, N and O.
- 2) All the circles are always apart from each with some even distance. The distance between any circles depends upon the radii of the circle. Suppose we have to find the distance between two innermost circles of radii r_1 and r_2 . Then distance between two innermost circles = $r_2 - r_1$. In the similar way other distances between any other concentric circles can be evaluated.
- 3) In the case of 5 concentric circles, the **tangent** drawn at any point on the inner circle is always the chord to the adjacent circle.

A **tangent** is an object, like a line, which touches a curve. The tangent only touches the curve at one point. That point is called the point of tangency.

Steiner's porism

Given two non-concentric circles with one inside the other, one can draw circles within the ringshaped region bounded by these two circles, touching one another successively and all

touching the original two circles. It may happen that the sequence of tangent circles closes so as to form a ring of n circles with the last touching the first. Steiner's porism is the result that gives the condition on n that such a chain is possible.

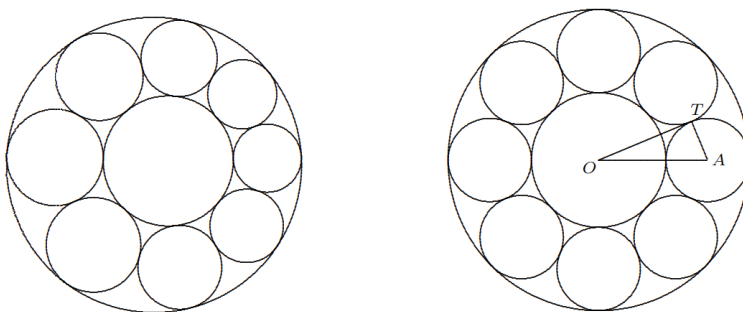


Figure 8: Steiner's Porism.

The solution is very simple. Simply invert the two non-intersecting circles into concentric ones as in the above figure on the right. Let the radii of the outer and inner concentric circles in this figure be a and b respectively. Then $OT A$ is a right-angled triangle with

$$OA = (a + b)/2, \text{ and } AT = (a - b)/2.$$

Suppose it is possible to inscribe n small circles between these two concentric circles.

By symmetry, all these small circles are congruent. Then $\angle T O A = \frac{\pi}{n}$. Thus

$$\sin \frac{\pi}{n} = \frac{AT}{OA} = \frac{a - b}{a + b} = \frac{a/b - 1}{a/b + 1},$$

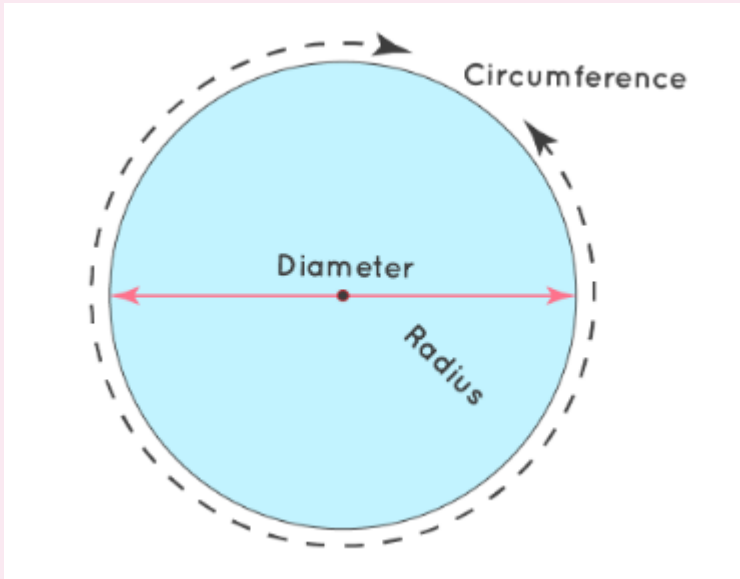
or

$$\frac{a}{b} = \frac{1 + \sin \frac{\pi}{n}}{1 - \sin \frac{\pi}{n}}.$$

When $n = 8$, the result is shown in the above figure with $a/b = 2.24$. When $n = 4$, $a/b = (\sqrt{2} + 1)^2$. In this case, there are 6 circles, each touching four others.

CONCEPT: HOW TO FIND THE CIRCUMFERENCE OF CIRCLE?

Although the circumference of a circle is the length of its boundary, it cannot be calculated with the help of a ruler (scale) like it is usually done for other polygons. This is because a circle is a curved figure. Therefore, to calculate the circumference of a circle, we apply a formula that uses the radius or the diameter of the circle and the value of Pi (π).



Pi is a special mathematical constant with a value approximated to 3.14159 or $\pi = 22/7$. The value of $\pi = 22/7$ is used in various formulas. It is the ratio of circumference to diameter, where $C = \pi D$. Let us consider a practical illustration to understand how to calculate the circumference of a circle with the help of the circumference formula

SUMMARY

- Orthogonality a property synonymous with perpendicularity when applied to vectors but applicable more generally to functions. Two elements of an inner product space are orthogonal when their inner product—for vectors, the dot product; for functions, the definite integral of their product—is zero.
- Inversion is the process of transforming points P to a corresponding set of points known as their inverse points.
- The Poincaré disc (in 2D) is the area inside a circle. The circle is not included. When using the Poincaré disc model, only points in the Poincaré disc are considered. The Poincaré disc makes up the entire world.
- A geodesic through points A and B is defined as the circular arc through A and B that is perpendicular to C^∞ . If A and B lie on the diameter of C^∞ , that diameter is the geodesic through A and B . A geodesic is the hyperbolic equivalence of a Euclidean line.
- A circle is a set of all points that are at an equal distance from a fixed point in a given plane. The fixed point is termed as the center of the circle and the common equal distance is known as the radius, the plural of which is radii.
- Concentric circles mean all those circles which have common center but different radii. It is interesting to note that there can be n number of concentric circles. The value of n can approach to infinite or we can say it has infinite value.

MULTIPLE CHOICE QUESTIONS

1. **In inversive geometry, what is the locus of points that are not changed by the inversion?**
 - a. The inversion circle.
 - b. The center of inversion.
 - c. A straight line.
 - d. The line at infinity.
2. **In inversive geometry, what happens to a line that does not pass through the center of inversion when it is inverted?**
 - a. It remains the same.
 - b. It becomes a circle.
 - c. It becomes a point.
 - d. It becomes another line.
3. **What is the cross ratio of four points A, B, C, and D under inversion?**
 - a. 1.
 - b. -1.
 - c. 0.
 - d. It depends on the positions of the points.
4. **When the center of inversion is located on the line of the four points, what happens to the cross ratio of those points?**
 - a. It becomes 0.
 - b. It becomes 1.
 - c. It remains the same.
 - d. It becomes undefined.
5. **In inversive geometry, what is the relation between the angles formed by two intersecting circles and their inverses?**
 - a. The angles are equal.
 - b. The angles are supplementary.
 - c. The angles are complementary.
 - d. The angles are halved.

Answer

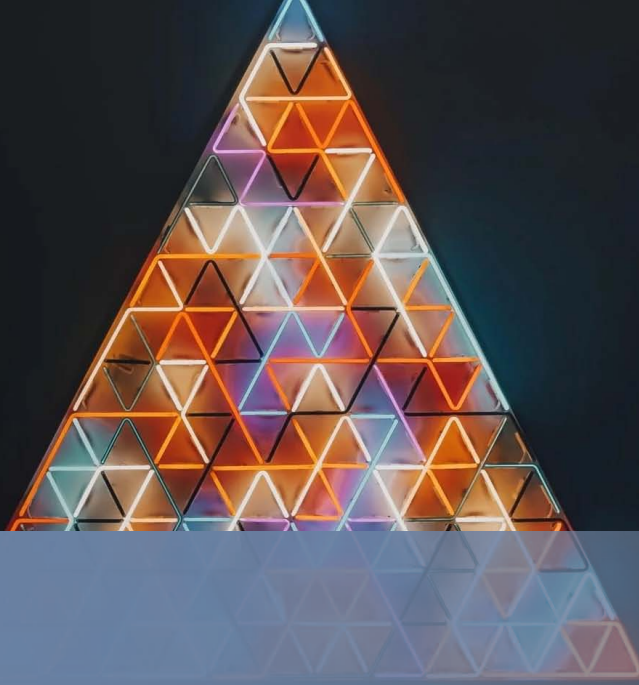
1. (a) 2. (b) 3. (b) 4. (b) 5. (a)

REVIEW QUESTIONS

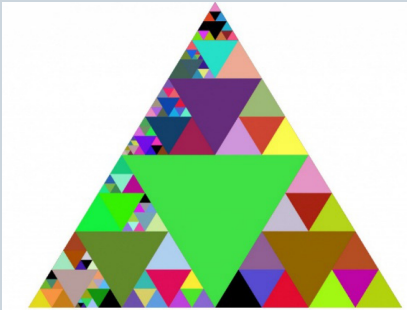
1. Let P and Q be two points which are not inverse of each other with respect to the circle ω . Give a construction of the circle through P and Q orthogonal to ω .
2. Prove that any two circles can be inverted into congruent circles.
3. A fountain which is in concentric shape has two radius one is 5 m and the other is 700 m. calculate the area between the two circles.
4. Write down the detailed procedure to draw the two concentric circles of radii 2 cm and 4 cm.
5. If circumferences of the outer circle and inner circle are 150 m and 100 m respectively. Find the width of the ring?
6. Given 4 circles touching one another to form a chain. Show that the 4 tangency points lie on a circle.
7. Find the area that is enclosed by two concentric circles whose radii are 2 cm and 5 cm.
8. Find the radius of the outer circle when chord of the outer circle which is the tangent to the inner circle measures 8 cm and the radius of the inner circle is 3 cm.

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TRIANGLES



Inversive geometry is the study of those properties of figures that are preserved by a generalization of a type of transformation of the Euclidean plane, called inversion. These transformations preserve angles and map generalized circles into generalized circles, where a generalized circle means either a circle or a line (loosely speaking, a circle with infinite radius). Many difficult problems in geometry become much more tractable when an inversion is applied.

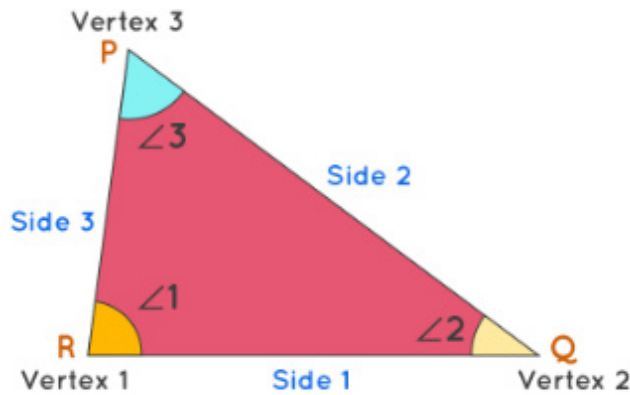
After studying this chapter, you will understand the core concepts of:

- Orthogonality
- Inversion
- Concentric Circles

INTRODUCTION

A triangle is a fundamental and timeless geometric shape that lies at the heart of geometry and mathematics. A triangle is a simple polygon with 3 sides and 3 interior angles. It is one of the basic shapes in geometry in which the 3 vertices are joined with each other and it is denoted by the symbol Δ . There are various types of triangles that are classified on the basis of the sides and angles.

A triangle consists of various parts. It has 3 angles, 3 sides, and 3 vertices. Observe the triangle PQR given below which shows the sides, the vertices and the interior angles of a triangle.



In the triangle given above:

- The three angles are, $\angle PQR$, $\angle QRP$, and $\angle RPQ$.
- The three sides are side PQ, side QR, and side RP.
- The three vertices are P, Q, and R

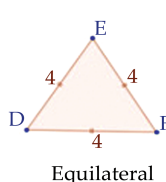
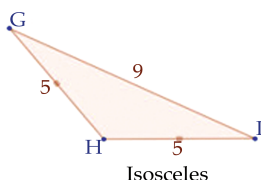
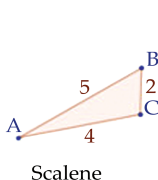


CORE CONCEPT: BASIC THEOREMS FOR TRIANGLES

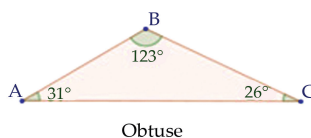
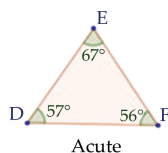
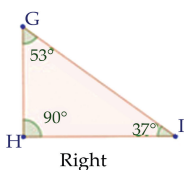
A triangle has three sides and is made of straight lines. A triangle may be classified by how many of its sides are of equal length. Or, it may be classified by what kind of angles it has.

Recall that a triangle is a shape with exactly three sides. Triangles can be classified by their sides and by their angles. When classifying a triangle by its sides, you should look to see if any of the sides are the same length. If no sides are the same length, then it is a **scalene triangle**. If two sides are the same length, then it is an isosceles triangle. If all three sides are the same length, then it is an equilateral triangle.

Scalene triangle has all its sides of different lengths.



When classifying a triangle by its angles, you should look at the size of the angles. If there is a right angle, then it is a right triangle. If the measures of all angles are less than 90° , then it is an acute triangle. If the measure of one angle is greater than 90° , then it is an obtuse triangle. Additionally, if all angles of a triangle are the same, the triangle is equiangular.



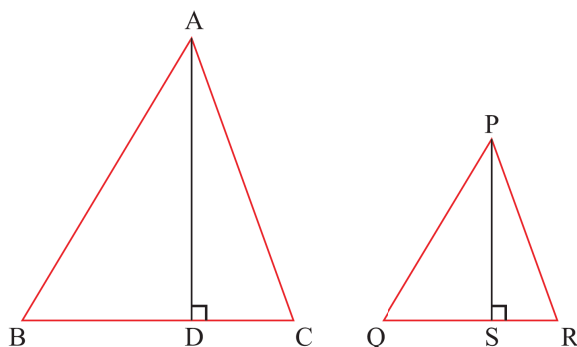
Triangle Inequalities

Triangles are governed by two important inequalities. The first is often referred to as the triangle inequality. It states that the length of a side of a triangle is always less than the sum of the lengths of the other two sides. Can you see why this must be true? Were one side of a triangle longer than the sum of the

Theorem is a statement that has been proven on the basis of previously established statements, such as other theorems, and generally accepted statements, such as axioms.

lengths of the other two, the triangle could not exist. As one side grows, the other two collapse toward that side until the altitude from the vertex opposite the growing side eventually becomes zero. This (an altitude of zero) would happen if the length of the one side was equal to the sum of the lengths of the other two. For The length of any side must be less than the sum of the lengths of the other sides.

The second inequality involving triangles has to do with opposite angles and sides. It states that when a pair of angles are unequal, the sides opposite them are also unequal. The converse is true also: when a pair of sides are unequal, so are their opposite angles. In essence, this theorem complements the **theorem** involving isosceles triangles, which stated that when sides or angles were equal, so were the sides or angles opposite them. The theorem about unequal pairs, though, goes a little farther.



Given unequal angles, the theorem holds that the longer side of the triangle will stand opposite the larger angle, and that the larger angle will stand opposite the longer side. This inequality is helpful to prove triangles *are not* congruent.

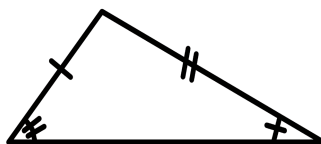
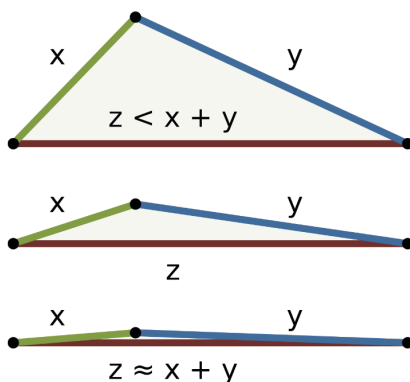


Figure 1: The larger of two unequal angles is opposite the longer of two unequal sides, and vice versa.

Notice the symbols in the figure above. When angles or sides are equal, the same number of tick marks, or small dashes, can be drawn on them. In a case where sides or angles are unequal, this can be symbolized by different numbers of tick marks on the angles or sides. More tick marks signifies a greater measure.



Exterior Angles of a Triangle

A triangle's exterior angle is just like that of any **polygon**; it is the angle created when one side of the triangle is extended past a vertex. The exterior angle has two interesting properties that follow from one another.

- 1) The exterior angle at a given vertex is equal in measure to the sum of the two remote interior angles. These remote interior angles are those at the other two vertices of the triangle.
- 2) Knowing this, it follows that the measure of any exterior angle is always greater than the measure of either remote interior angle. The first fact
 - (1), the equality, is useful for proving congruence;
 - (2), the inequality, is useful for disproving congruence.

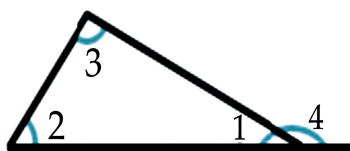


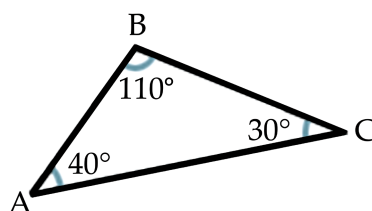
Figure 2: Angle 4 is greater than angle 2 and angle 3; angle 4 = angle 2 + angle 3.

Problem: Is it possible for the lengths of the sides of a triangle to be 1, 2, and 3? Why or why not?

No. $1 + 2 = 3$. The triangle inequality states that the sum of the lengths any two sides of a triangle must exceed the length of the third side.

Polygon is a plane figure that is bounded by a finite chain of straight line segments closing in a loop to form a closed polygonal chain or circuit.

Problem: Which side of the triangle below is the longest?



Side AC is the longest.

Problem: In triangle ABC, if side $AB = 3$, side $BC = 4$, and side $CA = 5$, which angle, A, B, or C, is the smallest?

Angle C

Problem: Is it possible for an exterior angle of a triangle to be smaller than one of the interior angles of the triangle?

Yes, the exterior angle must be acute, and the interior angle that is greater than the exterior angle must be adjacent to it. Only when the interior angle is obtuse is the exterior angle of a triangle smaller than that interior angle.

Problem: If an exterior angle of a triangle is 95 degrees, and one of the remote interior angles is 50 degrees, what is the measure of the other remote interior angle?

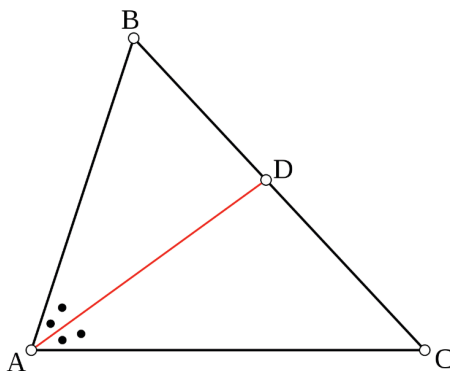
45 degrees

Bisection is the division of something into two equal or congruent parts, usually by a line, which is then called a bisector.

Theorems for Segments within Triangles

In learn properties of altitudes, medians, midsegments, angle bisectors, and perpendicular bisectors of triangles. All four of these types of lines or line segments within triangles are concurrent, meaning that the three medians of a triangle share intersecting points, as do the three altitudes, midsegments, angle bisectors, and perpendicular **bisectors**. The intersecting point is called the point of concurrency. The various points of concurrency for these four types of lines or line segments all have special properties.

This opposite side is called the base of the altitude, and the point where the altitude intersects the base (or its extension) is called the foot of the altitude.



Altitudes of a Triangle

The lines containing the altitudes of a triangle meet at one point called the orthocenter of the triangle. Because the orthocenter lies on the lines containing all three altitudes of a triangle, the segments joining the orthocenter to each side are perpendicular to the side. Keep in mind that the altitudes themselves aren't necessarily concurrent; the lines that contain the altitudes are the only guarantee. This means that the orthocenter isn't necessarily in the interior of the triangle.

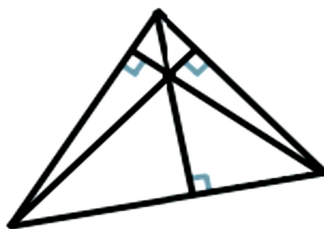
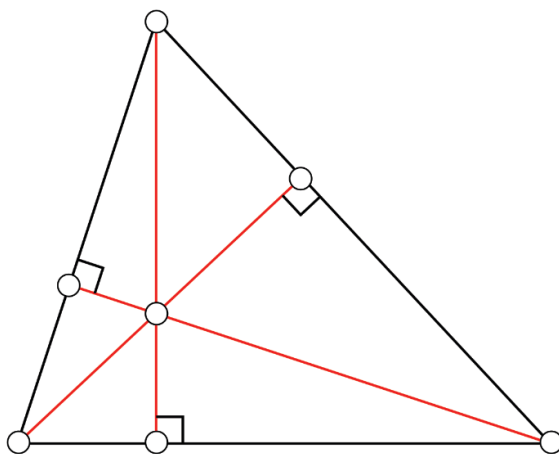


Figure 3: The lines containing the altitudes of a triangle and the orthocenter.

There are two other common theorems concerning altitudes of a triangle. Both concern the concept of similarity. The first states that the lengths of the altitudes of similar triangles follow the same proportions as the corresponding sides of the similar triangles.



The second states that the altitude of a right triangle drawn from the right angle to the hypotenuse divides the triangle into two similar triangles. These two triangles are also similar to the original triangle. The figure below illustrates this concept.

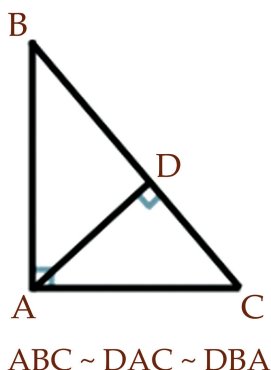


Figure 4: Triangles ABC, DAC, and DBA are similar to one another.

Medians of a Triangle

Every triangle has three medians, just like it has three altitudes, angle bisectors, and perpendicular bisectors. The medians of a triangle are the segments drawn from the vertices to the midpoints of the opposite sides. The point of intersection of all three medians is called the centroid of the triangle. The centroid of a triangle is twice as far from a given vertex than it is from the midpoint to which the median from that vertex goes. For example, if a median is drawn from vertex A to midpoint M through centroid C, the length of AC is twice the length of CM. The centroid is $\frac{2}{3}$ of the way from a given vertex to the opposite midpoint. The centroid is always on the interior of the triangle.

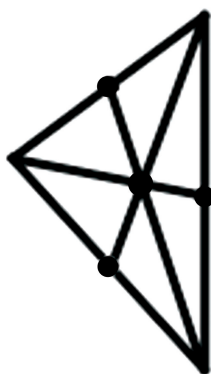
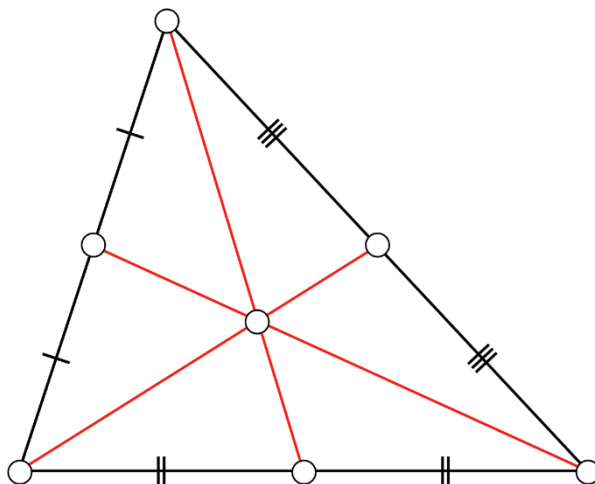


Figure 5: A triangle's medians and centroid.

Two more interesting things are true of medians.

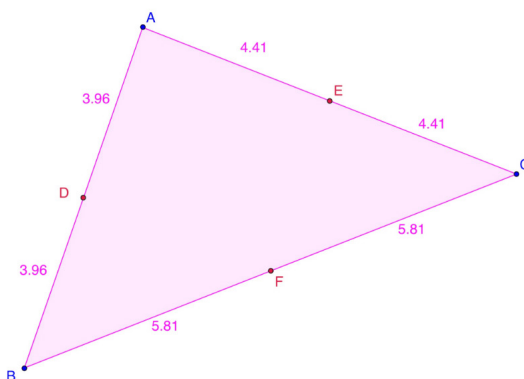
- 1) The lengths of the medians of similar triangles are of the same proportion as the lengths of corresponding sides.
- 2) The median of a right triangle from the right angle to the hypotenuse is half the length of the hypotenuse.



Theorems for Segments within Triangles

Midsegments of a Triangle

The midsegment of a triangle is a segment whose endpoints are both midpoints of sides. Every triangle has three midsegments. The midsegment of a triangle is always parallel to the third side (the side whose midpoint it doesn't include), and half as long as the third side.



Angle Bisectors of Triangles

The angle bisectors of a triangle intersect each other at a point called the in circle of the triangle. The in circle of a triangle is the same as the center of a circle inscribed in a triangle. Every triangle can have exactly one inscribed circle, whose center is the in circle of the triangle, which is the point at which the angle bisectors of the triangle intersect. The in circle, then, is equidistant from the three sides of the triangle--a property that results from the inherent congruency of the radii of a circle.

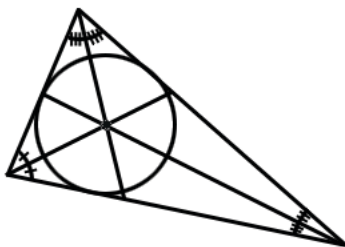
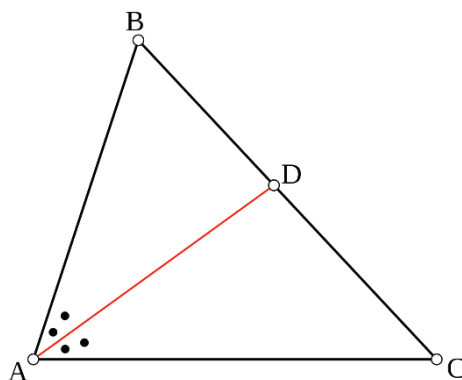


Figure 6: A triangle's angle bisectors and in circle.

Another property of angle bisectors has to do with the side opposite the bisected angle. An angle bisector divides the side opposite the bisected angle into two segments that are of the same proportion as the other two sides. For example, in triangle ABC above, let the angle at vertex A be bisected, and let the bisector intersect BC at point D. $BD/DC = BA/CA$.



Perpendicular Bisectors of Triangles

The three perpendicular bisectors of a triangle intersect at one point called the circumcenter of a triangle. The circumcenter is the center of the circle circumscribed about the triangle and is equidistant from all the vertices of the triangle. In this case the perpendicular bisectors of the sides of the triangles are lines, not segments. Therefore, the circumcenter of a triangle does not necessarily exist in the interior of the triangle. Often the perpendicular bisectors of a triangle intersect outside the triangle.

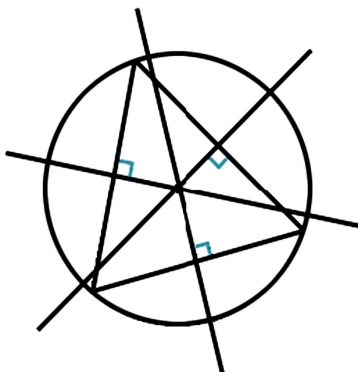
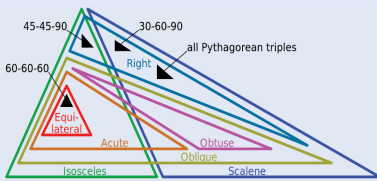
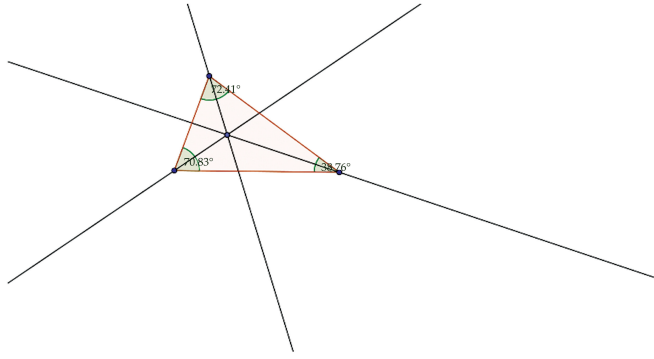


Figure 7: A triangle with its perpendicular bisectors and circumcenter.



CORE CONCEPT: TYPES OF TRIANGLES

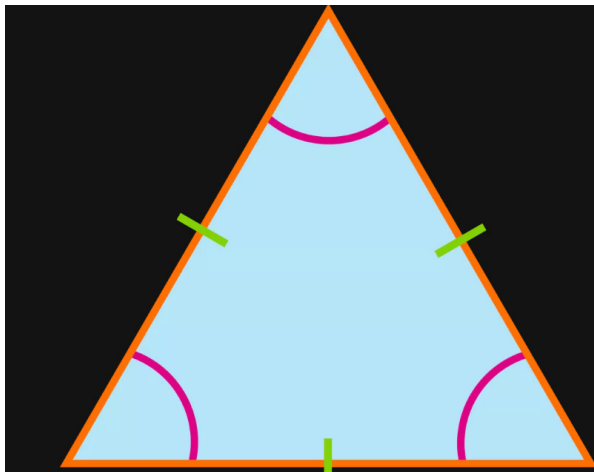
A triangle is a specific type of polygon having only three sides and three angle. As the name suggest, it tri-angle is a type of polygon having three angles. The sum of all interior angles of a triangle will always add up to 180 degrees.

Triangles can be broadly classified into two types, i.e.

- Triangle Based on the lengths of sides
- Triangle Based on interior angles

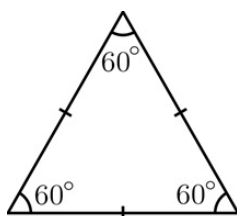
Types of Triangles Based on Sides

According to the lengths of their sides, triangles can be classified into three types:



Equilateral Triangle

In such a triangle, all the side lengths are equal. In such a case, each of the interior angles will have a measure of 60 degrees. The figure given below illustrates an equilateral triangle.



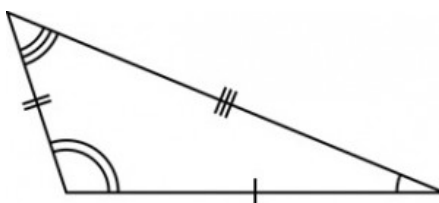
Isosceles Triangle

In an isosceles triangle, two of the three side lengths will be equal to each other. An isosceles triangle has two equal sides and two equal angles. The angles opposite the equal sides will be equal to each other. The figure given below illustrates an isosceles triangle.



Scalene Triangle

A scalene triangle has all side lengths of different measures. No side will be equal in length to any of the other sides in such a triangle. In a scalene triangle, all the interior angles will also be different. The figure given below illustrates a scalene triangle. You can see how none of the sides are equal in length.



Types of Triangles Based on Angles

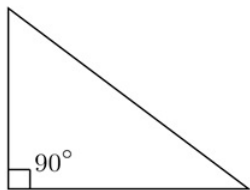
Triangles can be classified into three types with respect to their interior angles which are:

- Right-angled
- Acute-angled

- Obtuse-angled

Right Triangle

A right triangle is a triangle in which one of the angles is 90 degrees. In such a triangle, the side opposite to the right angle (90-degree angle) will be the longest side, and is called the hypotenuse. You may come across triangle types with combined names like right isosceles triangle and such, but this only implies that the triangle has two equal sides with one of the interior angles being 90 degrees. The figure given below illustrates a right triangle.



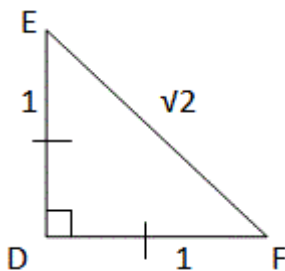
There are different types of right triangles. As of now, our focus is only on a special pair of **right triangles**.

- 45-45-90 triangle
- 30-60-90 triangle

45-45-90 triangle

A 45-45-90 triangle, as the name indicates, is a right triangle in which the other two angles are 45° each.

This is an isosceles right triangle.



In $\triangle DEF$, $DE = DF$ and $\angle D = 90^\circ$.

The sides in a 45-45-90 triangle are in the ratio $1: 1: \sqrt{2}$

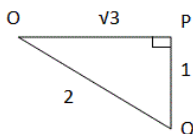
30-60-90 triangle

A 30-60-90 triangle, as the name indicates, is a right triangle in which the other two angles are 30° and 60°.

Hatch marks, also called tick marks, are used in diagrams of triangles and other geometric figures to identify sides of equal lengths. A side can be marked with a pattern of "ticks", short line segments in the form of tally marks; two sides have equal lengths if they are both marked with the same pattern.

Right triangles A central theorem is the Pythagorean Theorem, which states in any right triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the two other sides.

This is a scalene right triangle as none of the sides or angles are equal.

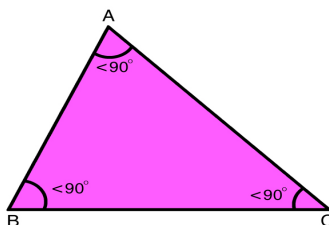


The sides in a 30-60-90 triangle are in the ratio $1:\sqrt{3}:2$

Like any other right triangle, these two triangles satisfy the Pythagorean Theorem.

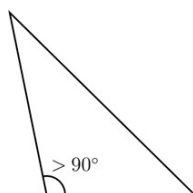
Acute Triangles

If a triangle has all three interior angles which are acute, then it is an acute triangle. In other words, if all interior angles are less than 90 degrees, then it is an acute triangle. The figure given below illustrates an acute triangle.



Obtuse Triangles

If a triangle is such that one of its three interior angles has a measure greater than 90 degrees, then it is an obtuse triangle. In other words, if one of the angles in a triangle is an obtuse angle, then the triangle is called an obtuse triangle or an obtuse angled triangle. The figure given below illustrates an obtuse triangle.

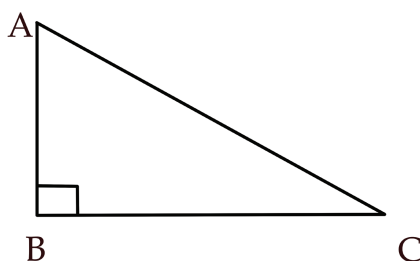


Basic properties of triangles

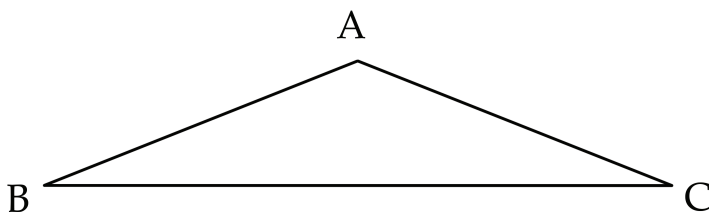
- The sum of the angles in a triangle is 180° . This is called the angle-sum property.
- The sum of the lengths of any two sides of a triangle is greater than the length of the third side. Similarly, the difference between the lengths of any two sides of a triangle is less than the length of the third side.

The sum of the lengths of any two sides of a triangle must be greater than the length of the third side. This condition is essential to determine if a set of side lengths can form a valid triangle.

- The side opposite to the largest angle is the longest side of the triangle and the side opposite to the smallest angle is the shortest side of the triangle.

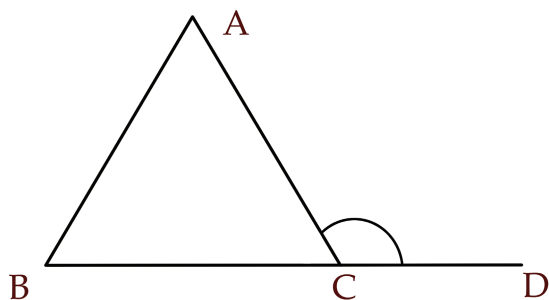


In the figure above, $\angle B$ is the largest angle and the side opposite to it (hypotenuse), is the largest side of the triangle.



In the figure above, $\angle A$ is the largest angle and the side opposite to it, BC is the largest side of the triangle.

- An exterior angle of a triangle is equal to the sum of its interior opposite angles. This is called the exterior angle property of a triangle.



Here, $\angle ACD$ is the exterior angle to the $\triangle ABC$.

According to the exterior angle property, $\angle ACD = \angle CAB + \angle ABC$.

Similarity and Congruency in Triangles

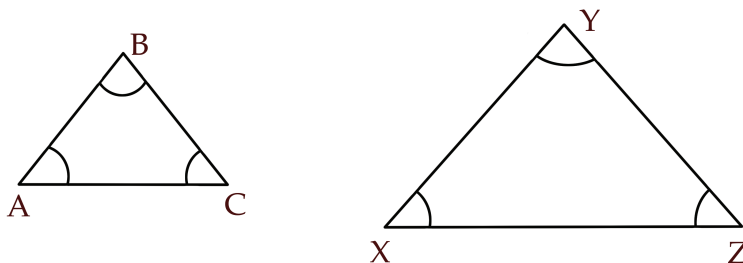
Figures with same size and shape are congruent figures. If two shapes are congruent, they remain congruent even if they are moved or rotated. The shapes would also remain congruent if we reflect the shapes by producing mirror images. Two geometrical shapes are congruent if they cover each other exactly.

Figures with same shape but with proportional sizes are similar figures. They remain similar even if they are moved or rotated.

Similarity of triangles

Two triangles are said to be similar if the corresponding angles of two triangles are congruent and lengths of corresponding sides are proportional.

It is written as $\Delta ABC \sim \Delta XYZ$ and said as ΔABC 'is similar to' ΔXYZ .



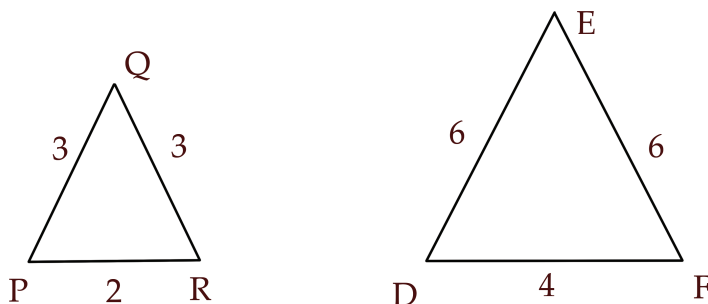
Here, $\angle A = \angle X$, $\angle B = \angle Y$ and $\angle C = \angle Z$ AND

$$AB / XY = BC / YZ = CA / ZX$$

The necessary and sufficient conditions for two triangles to be similar are as follows:

(1) Side-Side-Side (SSS) criterion for similarity:

If three sides of a triangle are proportional to the corresponding three sides of another triangle then the triangles are said to be similar.

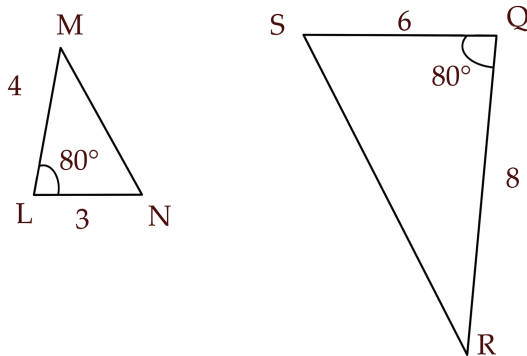


Here, $\Delta PQR \sim \Delta DEF$ as

$$PQ / DE = QR / EF = RP / FD$$

(2) Side-Angle-Side (SAS) criterion for similarity:

If the corresponding two sides of the two triangles are proportional and one included angle is equal to the corresponding included angle of another triangle then the triangles are similar.



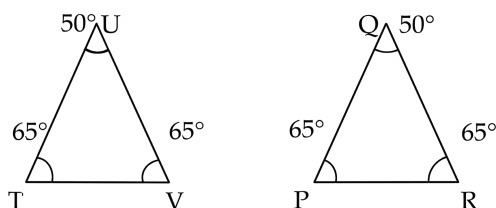
Here, $\Delta LMN \sim \Delta QRS$ in which

$$\angle L = \angle Q$$

$$QS / LN = QR / LM$$

(3) Angle-Angle-Angle (AAA) criterion for similarity:

If the three corresponding angles of the two triangles are equal then the two triangles are similar.



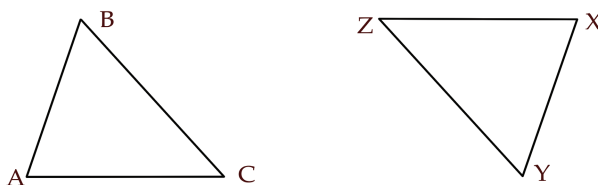
Here $\Delta TUV \sim \Delta PQR$ as

$$\angle T = \angle P, \angle U = \angle Q \text{ and } \angle V = \angle R$$

Congruency of triangles

Two triangles are said to be congruent if all the sides of one triangle are equal to the corresponding sides of another triangle and the corresponding angles are equal.

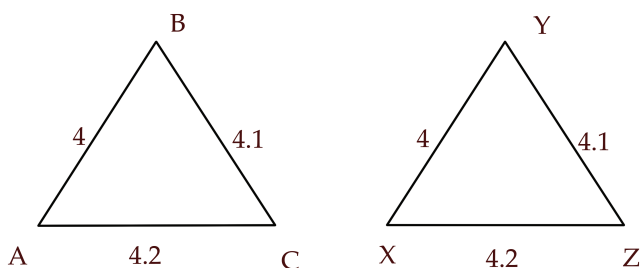
It is written as $\Delta ABC \cong \Delta XYZ$ and said as ΔABC 'is congruent to' ΔXYZ .



The necessary and sufficient conditions for two triangles to be congruent are as follows:

(1) Side-Side-Side (SSS) criterion for congruence:

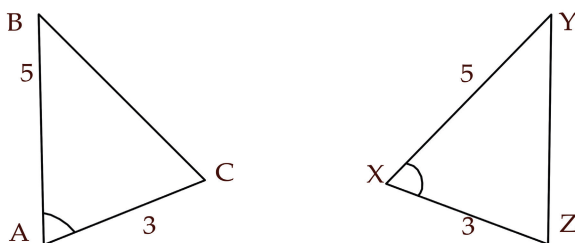
If three sides of a triangle are equal to the corresponding three sides of another triangle then the triangles are said to be congruent.



Here, $\triangle ABC \cong \triangle XYZ$ as $AB = XY$, $BC = YZ$ and $AC = XZ$.

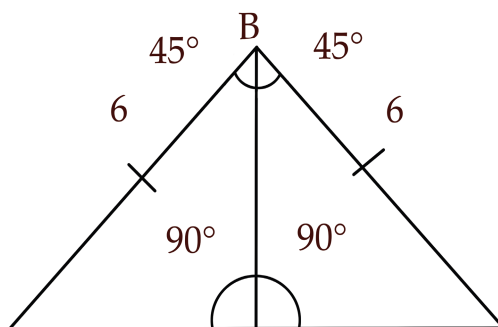
(2) Side-Angle-Side (SAS) criterion for congruence:

If two sides and the angle included between the two sides of a triangle are equal to the corresponding two sides and the included angle of another triangle, then the triangles are congruent.



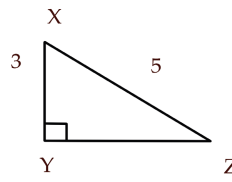
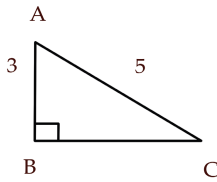
Here, $\triangle ABC \cong \triangle XYZ$ as $AB = XY$, $\angle A = \angle X$ and $AC = XZ$.

(3) Angle-Side-Angle (ASA) criterion for congruence: If two angles and the included side of a triangle are equal to the corresponding two angles and the included side of another triangle then the triangles are congruent.



In the figure above, $\triangle ABD \cong \triangle CBD$ in which $\angle ABD = \angle CBD$, $AB = CB$ and $\angle ADB = \angle CDB$.

4) Right-Angle Hypotenuse criterion of congruence: If the hypotenuse and one side of a right-angled triangle are equal to the corresponding hypotenuse and side of another right-angled triangle, then the triangles are congruent.

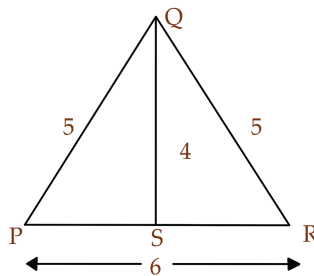


Here, $\angle B = \angle Y = 90^\circ$ and $AB = XY$, $AC = XZ$.

Area of a triangle:

The Area of a triangle is given by the formula

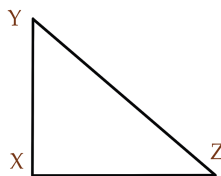
Area of a triangle = $(1/2) * \text{Base} * \text{Height}$



To find the area of a triangle, we draw a **perpendicular** line from the base to the opposite vertex which gives the height of the triangle.

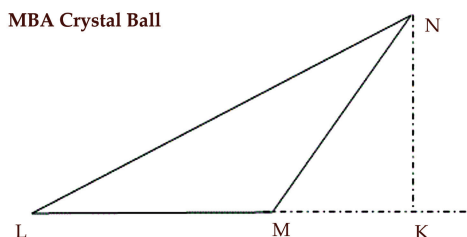
So the area of the $\Delta PQR = (1/2) * (PR * QS) = (1/2) * 6 * 4 = 12$ sq. units.

For a right triangle, it's easy to find the area as there is a side perpendicular to the base, so we can consider it as height.



The height of the ΔXYZ is XY and its area is $(1/2) * XZ * XY$ sq. units.

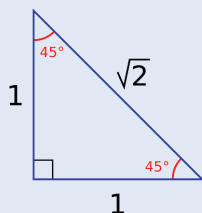
Now, how do we find the area of an obtuse triangle LMN?



Perpendicular is the relationship between two lines which meet at a right angle (90 degrees).

For an obtuse triangle, we extend the base and draw a line perpendicular from the vertex to the extended base which becomes the height of the triangle.

Hence, the area of the $\Delta LMN = (1/2) * LM * NK$ sq. units.



CORE CONCEPT: TRIANGLE FORMULAE

We now will use the right-triangle trig formulas to find the areas of right triangles, equilateral triangles, and isosceles triangles. We then will use the isosceles triangles to find the area of regular n -sided polygons. Finally, we will use Heron's Formula to find the areas of other scalene triangles.

Consider a triangle such as that shown in Figure 8.

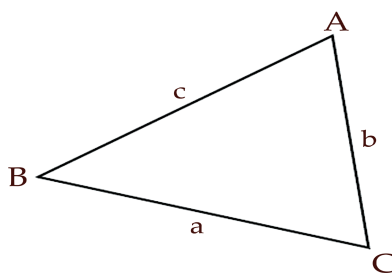


Figure 8. A triangle with six pieces of information: angles at A , B , and C ; sides a , b and c .

There are six pieces of information available: angles at A , B and C , and the sides a , b and c . The angle at A is usually written A , and so on. Notice that we label the sides according to the following convention:

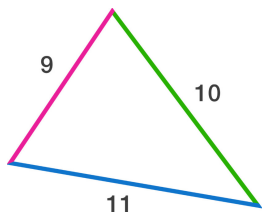
Side b is opposite the angle B

Side c is opposite the angle C

Side a is opposite the angle A

Now if we take three of these six pieces of information we will (except in two special cases) be able to draw a unique triangle.

Let's deal first with the special cases.



The first special case

The first special case is when we know just the three angles. Then having drawn one triangle with these angles, we can draw as many more triangles as we wish, all with the same shape as the original, but larger or smaller. All will have the same angles but the sizes of the triangles will be different. We cannot define a unique triangle when we know just the three angles. This behavior is illustrated in Figure 9 where the corresponding angles in the two triangles are the same, but clearly the triangles are of different sizes.

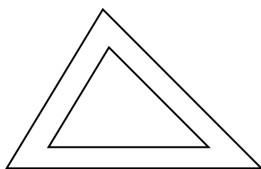


Figure 9. Given just the three angles we cannot construct a unique triangle.

The second special case

There is a second special case whereby if we are given three pieces of information it is impossible to construct a unique triangle. Suppose we are given one angle, A say, and the lengths of two of the sides. This situation is illustrated in Figure 10 (a). The first given side is marked //. The second given side is marked /; this can be placed in two different locations as shown in Figures 10(b) and 3c). Consequently it is impossible to construct a unique triangle.

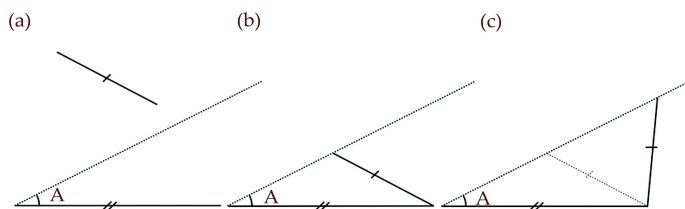
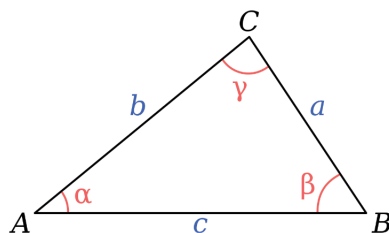
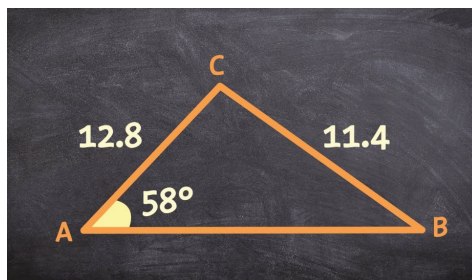


Figure 10. It is impossible to draw a unique triangle given one angle and two side **lengths**

Length is the most extended dimension of an object.



Apart from these two special cases, if we are given three pieces of information about the triangle we will be able to draw it uniquely. There are formulae for doing this which we describe in the following sections.



The cosine formulae

We can use the cosine formulae when three sides of the triangle are given.

If we consider the formula $c^2 = a^2 + b^2 - 2ab \cos C$, and refer to Figure 11 we note that we can use it to find side c when we are given two sides (a and b) and the included angle C .

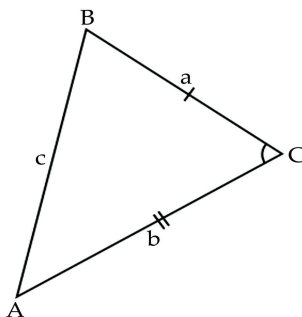
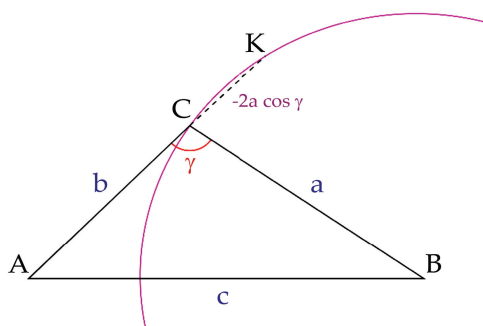


Figure 11. Using the cosine formulae to find c if we know sides a and b and the included angle C .

Similar observations can be made of the other two formulae.

So there are in fact six cosine formulae, one for each of the angles - that's three altogether, and one for each of the sides, that's another three. We only need to learn two of them, one for the angle, one for the side and then just cycle the letters through to find the others.



The sine formulae

We can use the sine formulae to find a side, given two sides and an angle which is NOT included between the two given sides.

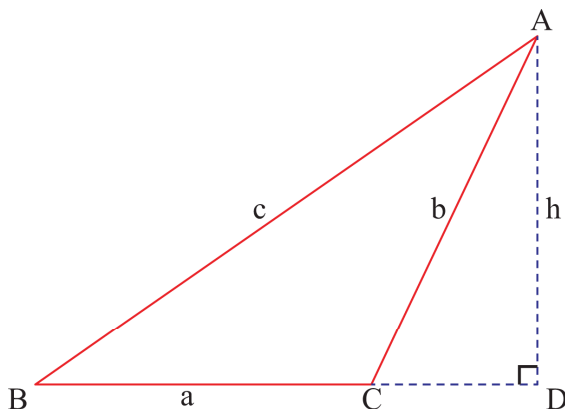
R is the radius of the circumcircle - the **circumcircle** is the circle that we can draw that will go through all the points of the triangle as shown in Figure 12.

Circumcircle is a circle which passes through all the vertices of the polygon.

Taking just the first three terms in the formulae we can rearrange them to give

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

and we can use the formulae in this form as well.



Some examples of the use of the cosine and sine formulae

Example

Suppose we are given all three sides of a triangle:

$$a = 5, \quad b = 7, \quad c = 10$$

We will use this information to determine angle A using the cosine formula:

$$\begin{aligned}
 \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
 &= \frac{7^2 + 10^2 - 5^2}{2 \times 7 \times 10} \\
 &= \frac{49 + 100 - 25}{140} \\
 &= \frac{124}{140} \\
 A &= \cos^{-1} \frac{124}{140} = 27.7^\circ \quad (1 \text{ d.p.})
 \end{aligned}$$

The remaining angles can be found by applying the other cosine formulae.

Example

Suppose we are given two sides of a triangle and an angle, as follows

$$b = 10, \quad c = 5, \quad A = 120^\circ$$

It's not immediately obvious what information we have been given. In the last Example it was very clear. So we make a sketch to mark out the information we have been given as shown in Figure 12.

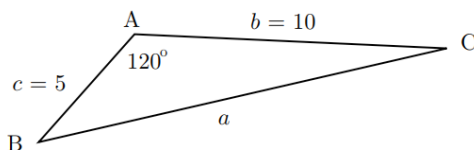


Figure 12. The information given in the Example.

From the Figure we can deduce that we have been given 2 sides and the included angle. We can use the cosine formula to deduce the length of side a.

$$\begin{aligned}
 a^2 &= b^2 + c^2 - 2bc \cos A \\
 &= 10^2 + 5^2 - 2 \times 10 \times 5 \cos 120^\circ \\
 &= 100 + 25 - 100 \cos 120^\circ \\
 &= 125 - 100 \times \left(-\frac{1}{2}\right) \\
 &= 175 \\
 a &= \sqrt{175} = 13.2 \quad (3 \text{ s.f.})
 \end{aligned}$$

Now that we have worked out the length of side a, we have three sides. We could use the cosine formulae to find out either one of the remaining angles.

Example

Suppose we are given the following information:

$$c = 8, \quad b = 12, \quad C = 30^\circ$$

Note that we are given two side lengths and an angle which is not the included angle. Referring back to the special cases described in the Introduction you will see that with this information there is the possibility that we can obtain two distinct triangles with this information.

As before we need a sketch in order to understand the information (Figure 13.)

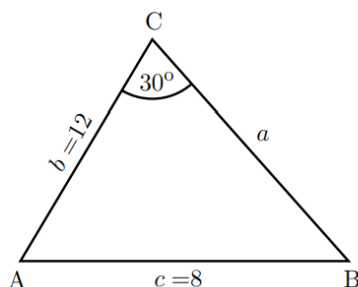


Figure 13. We are given two sides and a non-included angle.

Because we have been given two sides and a non-included angle we use the sine formulae.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Because we are given b, c and C we use the following part of the formula in order to find angle B.

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin C}{c} \\ \frac{\sin B}{12} &= \frac{\sin 30^\circ}{8} \\ \sin B &= \frac{12 \times \sin 30^\circ}{8} \\ &= \frac{12 \times \frac{1}{2}}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

$$= 0.75$$

$$B = \sin^{-1} 0.75$$

$$= 48.6^\circ \quad (1 \text{ d.p.})$$

Now there is a potential complication here because there is another angle with sine equal to 0.75. Specifically, B could equal $180^\circ - 48.6^\circ = 131.4^\circ$.

In the first case the angles of the triangle are then:

$$C = 30^\circ, \quad B = 48.6^\circ, \quad A = 180^\circ - 78.6^\circ = 101.4^\circ$$

In the second case we have:

$$C = 30^\circ, \quad B = 131.4^\circ, \quad A = 180^\circ - 161.4^\circ = 18.6^\circ.$$

The situation is depicted in Figure 14. In order to solve the triangle completely we must deal with the two cases separately in order to find the remaining unknown a.

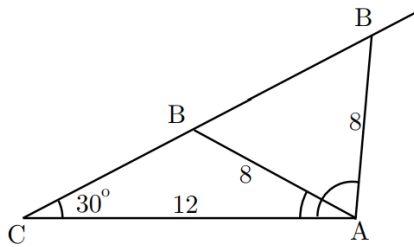


Figure 14. There are two possible triangles.

Case 1. Here $C = 30^\circ$, $B = 48.6^\circ$, $A = 101.4^\circ$. We use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned} a &= \frac{12 \sin 101.4^\circ}{\sin 48.6^\circ} \\ &= 15.7 \quad (1 \text{ d.p.}) \end{aligned}$$

Case 2. Here $C = 30^\circ$, $B = 131.4^\circ$, $A = 18.6^\circ$. Again we can use the sine rule in the form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\begin{aligned}
 a &= \frac{12 \sin 18.6^\circ}{\sin 131.4^\circ} \\
 &= 5.1 \quad (1 \text{ d.p.})
 \end{aligned}$$

The area of a triangle

We now look at a set of formulae which will give us the area of a triangle. A standard formula is

$$\text{area} = \frac{1}{2} \times \text{base} \times \text{height}$$

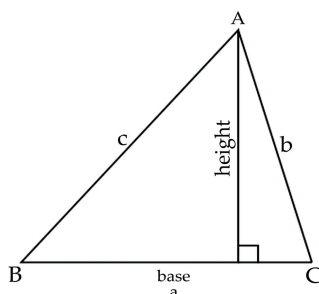


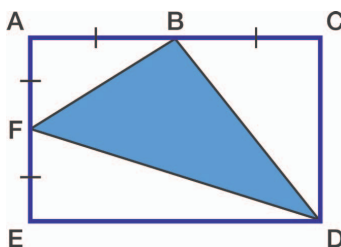
Figure 15. The area of the triangle is $\frac{1}{2} \times \text{base} \times \text{height}$.

Let us assume we know the lengths a , b and c , and the angle at B . Consider the right-angled triangle on the left-hand side of Figure 15. In this triangle

$$\sin B = \frac{\text{height}}{c}$$

and so, by rearranging,

$$\text{height} = c \sin B$$



Then from the formula for the area of the large triangle, $\triangle ABC$

$$\begin{aligned}
 \text{area} &= \frac{1}{2} \times \text{base} \times \text{height} \\
 &= \frac{1}{2} ac \sin B
 \end{aligned}$$

Now consider the right-angled triangle on the right-hand side in Figure 15.

$$\sin C = \frac{\text{height}}{b}$$

and so, by rearranging,

$$\text{height} = b \sin C$$

So, the area of the large triangle, $\triangle ABC$, is also given by

$$\text{area} = \frac{1}{2}ab \sin C$$

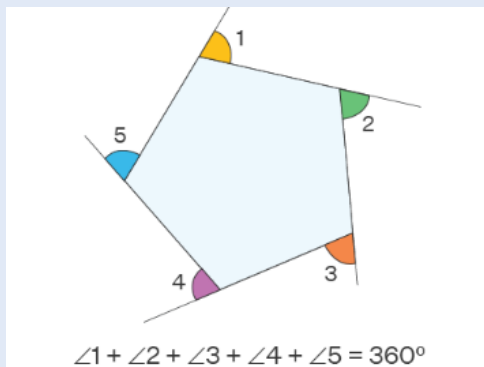
It is also possible to show that the formula

$$\text{area} = \frac{1}{2}bc \sin A$$

will also give the area of the large triangle.

CONCEPT: POLYGON ANGLE SUM THEOREM

The polygon exterior angle sum theorem states that “the sum of all exterior angles of a convex polygon is equal to 360° ”. Let’s consider the polygon given below.



We can observe that in this 5-sided polygon, the sum of all exterior angles is 360° by polygon angle sum theorem. The number of interior angles is equal to the number of sides. The measure of an interior angle of a regular polygon can be calculated using the formula, Interior angle = $180^\circ(n-2)/n$, where n is the number of sides. Each exterior angle of a regular polygon is equal and the sum of the exterior angles of a polygon is 360° . An exterior angle can be calculated using the formula, Exterior Angle = $360^\circ/n$, where n is the number of sides.

SUMMARY

- A triangle is a fundamental and timeless geometric shape that lies at the heart of geometry and mathematics. A triangle is a simple polygon with 3 sides and 3 interior angles. It is one of the basic shapes in geometry in which the 3 vertices are joined with each other and it is denoted by the symbol Δ .
- A triangle's exterior angle is just like that of any polygon; it is the angle created when one side of the triangle is extended past a vertex.
- The midsegment of a triangle is a segment whose endpoints are both midpoints of sides. Every triangle has three midsegments. The midsegment of a triangle is always parallel to the third side (the side whose midpoint it doesn't include), and half as long as the third side.
- The angle bisectors of a triangle intersect each other at a point called the in circle of the triangle. The in circle of a triangle is the same as the center of a circle inscribed in a triangle.
- In an isosceles triangle, two of the three side lengths will be equal to each other. An isosceles triangle has two equal sides and two equal angles. The angles opposite the equal sides will be equal to each other.
- A right triangle is a triangle in which one of the angles is 90 degrees. In such a triangle, the side opposite to the right angle (90-degree angle) will be the longest side, and is called the hypotenuse.

MULTIPLE CHOICE QUESTIONS

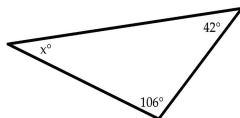
1. In triangle ABC, if $AB=BC$ and $\angle B = 70^\circ$, $\angle A$ will be:
 - a. 70°
 - b. 110°
 - c. 55°
 - d. 130°
2. For two triangles, if two angles and the included side of one triangle are equal to two angles and the included side of another triangle. Then the congruency rule is:
 - a. SSS
 - b. ASA
 - c. SAS
 - d. None of the above
3. A triangle in which two sides are equal is called:
 - a. Scalene triangle
 - b. Equilateral triangle
 - c. Isosceles triangle
 - d. None of the above
4. The angles opposite to equal sides of a triangle are:
 - a. Equal
 - b. Unequal
 - c. supplementary angles
 - d. Complementary angles
5. If E and F are the midpoints of equal sides AB and AC of a triangle ABC. Then:
 - a. $BF = AC$
 - b. $BF = AF$
 - c. $CE = AB$
 - d. $BF = CE$

Answer

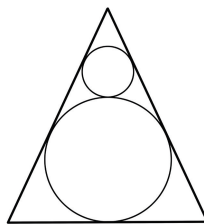
1. (c) 2. (b) 3. (c) 4. (a) 5. (d)

REVIEW QUESTIONS

1. What are triangles?
2. Find the value of x in the triangle shown below.



3. Two circles are placed in an equilateral triangle as shown in the figure. What is the ratio of the area of the smaller circle to that of the equilateral triangle?



4. The length of the hypotenuse of a right triangle is 40 cm. The size of one of its angles is 45 degrees. What are the exact lengths of the other two sides of the triangle?
5. Triangle ABC is an isosceles triangle. The length of the base is 20 meters and the corresponding height is 24 meters. Find the perimeter of ABC. (Round your answer to the nearest tenth of a meter).
6. A triangle has an area of 90 square cm. Find the length of the base if the corresponding base is 3 cm more than the height.
7. Determine the area of the triangle enclosed by the lines $y = -4$, $x = 1$ and $y = -2x + 8$:
8. Show that the triangle with vertices $A(-1,6)$, $B(2,6)$, $C(2,2)$ is a right triangle and find its area.

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BASICS OF POLYGON



A Polygon is a closed plane shape with three or more straight line segments connected. Polygon is a closed plane figure where line segments are joined together, the sides do not cross each other but they meet at every vertex.

After studying this chapter, you will understand the core concepts of:

- Polygon Properties
- Angles of Polygons
-

INTRODUCTION

Polygon is a fundamental geometric concept that plays a significant role in mathematics and various fields of science and engineering. A polygon is a two-dimensional closed shape consisting of straight sides and vertices. These vertices are the points where the sides of the polygon intersect. The term “polygon” originates from the Greek words “poly,” meaning many, and “gonia,” meaning angle, reflecting its characteristic feature of having multiple angles.

Polygons come in various forms, sizes, and complexity. They are classified based on the number of sides they possess, ranging from triangles (3 sides) to quadrilaterals (4 sides), pentagons (5 sides), hexagons (6 sides), heptagons (7 sides), and so on. Regular polygons have equal sides and equal interior angles, while irregular polygons have sides and angles of varying lengths.

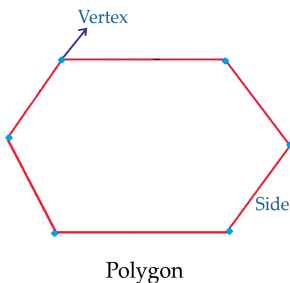
Understanding the properties of polygons is essential in geometry. For instance, the interior angles of a polygon can be calculated using the formula $(n-2) * 180^\circ$, where n is the number of sides. This formula provides a general method to determine the sum of interior angles for any polygon. Additionally, the perimeter of a polygon is the sum of its side lengths, and its area can be computed using various formulas, depending on the type of polygon.



CORE CONCEPT: POLYGON PROPERTIES

A polygon is any 2-dimensional shape formed with straight lines. Triangles, quadrilaterals, pentagons, and hexagons are all examples of polygons. The name tells you how many sides the shape has. For example, a triangle has three sides, and a quadrilateral has four sides.

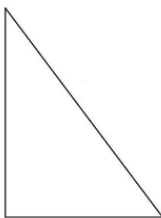
A polygon is a 2D shape which is made up of straight line segments. A regular polygon has all sides and angles equal (e.g. a square is a regular quadrilateral). A triangle with all sides and angles equal is known as an equilateral triangle.



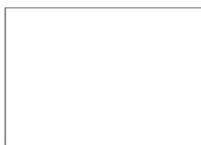
A number of coplanar line segments which are connected end to end to form a closed shape is called as a polygon. Examples of polygons are triangles, quadrilateral, pentagons, and hexagons and so on. There are a number of properties of polygon which make them unique from other geometrical shapes.

Polygons are named after the number of sides they have.

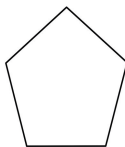
Three Sides – Triangle



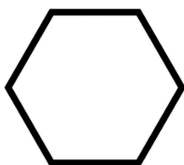
Four Sides – Quadrilateral (e.g. square or rectangle)



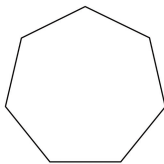
Five Sides – Pentagon



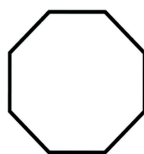
Six Sides – Hexagon



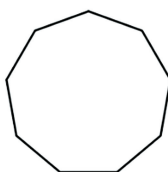
Seven Sides -Heptagon



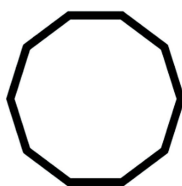
Eight Sides – Octagon



Nine Sides – Nonagon



Ten Sides – Decagon



Polygon Parts

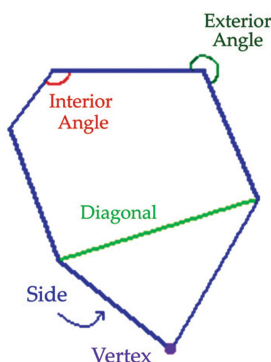
Line segment is a part of a line that is bounded by two distinct end points, and contains every point on the line between its endpoints.

A closed plane figure bounded by three or more **line segments** is called a polygon. The polygon is a Greek word in which the meaning of ‘poly’ is ‘many’ and the meaning of the ‘gon’ is ‘angle’.

There are a number of parts of a polygon. The parts of the polygon is the same for all types of polygons. The different parts of the polygon are,

- **Side** - one of the line segments that make up the polygon.
- **Vertex** - point where two sides meet. Two or more of these points are called vertices.
- **Diagonal** - a line connecting two vertices that is not a side.
- **Interior Angle** - Angle formed by two adjacent sides inside the polygon.

- *Exterior Angle* - Angle formed by two adjacent sides outside the polygon.



Types of Polygon

A polygon is a plane shape that has at least three sides. Etymologically, they have multiple (poly) angles (gonia). In other words, a two-dimensional object which is surrounded by finite number of straight lines binding in the form of a closed loop, is categorized as a polygon. The line segments from which polygon is made, are known as polygon's sides or edges. While, a point or the corner where any two sides meet, is called a vertex (plural vertices) of the polygon. Also, the interior part of the polygon is said to be its body. There are a number of two-dimensional shapes that have been classified as polygons.

Polygons are categorized into different types based on the number of sides as well as the measure of the angles. Different types of polygons:

- Regular Polygons
- Irregular Polygons
- Concave Polygons
- Convex Polygons
- Quadrilateral Polygons
- Pentagon Polygons

Regular Polygons

A polygon is called regular if it has equal sides and angles. Thus, a regular triangle is an equilateral triangle, and a regular quadrilateral is a square. A general problem since antiquity has been the problem of constructing a regular n -gon, for different n , with only ruler and compass. For example, Euclid constructed a regular pentagon by applying the above-mentioned five important theorems in an ingenious combination.

Techniques, such as bisecting the angles of known constructions, exist for constructing regular n -gons for many values, but none is known for the general case. In 1797, following centuries without any progress, Gauss surprised the

mathematical community by discovering a construction for the 17-gon. More generally, Gauss was able to show that for a prime number p , the regular p -gon is constructible if and only if p is a “Fermat prime”: $p = F(k) = 2^{2^k} + 1$. Because it is not known in general which $F(k)$ are prime, the construction problem for regular n -gons is still open.

Three other unsolved construction problems from antiquity were finally settled in the 19th century by applying tools not available to the Greeks. Comparatively simple algebraic methods showed that it is not possible to trisect an angle with ruler and compass or to construct a cube with a volume double that of a given cube.

Regular polygons may be either convex or star. In the limit, a sequence of regular polygons with an increasing number of sides approximates a circle, if the perimeter is fixed, or a regular apeirogon, if the edge length is fixed.

Properties of a Regular Polygon

There are a number of properties of polygon. They are as listed follows:

- All the sides of a regular polygon are equal.
- All the angles of a regular polygon are equal.

- The internal angle of a regular polygon with n sides is $\frac{n-2}{n} 180^\circ$. For Example, the interior angle of a regular Pentagon (5 sides) is, $\frac{5-2}{5} 180^\circ = 108^\circ$.

- Both the External Angle and the Internal Angle are taken from the same line. And so, they add up to 180° .
- The external angle of a regular polygon is $180^\circ - \text{Internal Angle}$. For Example, The internal angle of a pentagon is 108° . And so, the external angle of a pentagon is $180^\circ - 108^\circ = 72^\circ$.
- All the polygon have diagonals which is a line that connects one corner to another corner, but not the sides of the polygon.

- The number of diagonals of a polygon is equal to $\frac{n(n-3)}{2}$. For Example, the number of diagonals of a Square is 2. A square has four corner. So, $\frac{4(4-3)}{2} = 2$.
- A circle formed outside a polygon connecting all the vertices of a Polygon is called as a circumcircle.
- A **circle** formed inside a polygon connecting all the sides of a Polygon and so each of the sides is a tangent to the incircle is called as a incircle.
- The radius of a circumcircle is called as the radius of the polygon.

- The radius of an incircle is called as an apothem of the polygon.

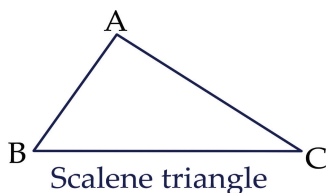
Irregular Polygons

An irregular polygon is a polygon that is not regular. That is, an irregular polygon is a polygon that does not have all sides of equal lengths and all interior angles equal. For example, consider this picture of an SD card: Notice the SD card has 5 sides and they are not all equal length. Also, the interior angles are not all equal. Therefore, the SD card takes on the shape of an irregular polygon.

These are all things that you may come across on a daily basis. Isn't it neat how mathematics can make us look at the world around us so differently when we learn something new?

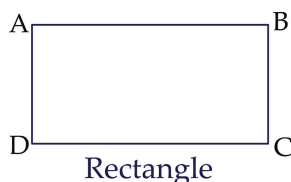
Examples of irregular polygon:

In the adjoining figure of a scalene triangle ABC there are three sides i.e., AB, BC and CA are unequal and there are three angles i.e., $\angle ABC$, $\angle BCA$ and $\angle CAB$ are unequal.



Therefore, a scalene triangle is an irregular polygon.

In the adjoining figure of a rectangle ABCD there are four sides i.e., AB, BC, CD and DA where the opposite sides are equal i.e., $AB = CD$ and $BC = AD$. So, all the sides are not equal to each other.



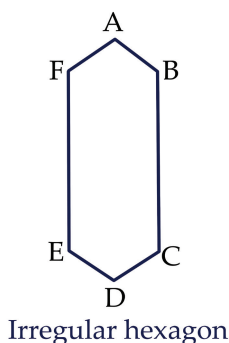
Similarly, among the four angles i.e., $\angle ABC$, $\angle BCD$, $\angle CDA$ and $\angle DAB$ where the opposite angles are equal i.e., $\angle ABC = \angle CDA$ and $\angle BCD = \angle DAB$. So, all the angles are not equal to each other.

Therefore, a square is an irregular polygon.

In the adjoining figure of an irregular hexagon ABCDEF

Circle is a simple closed shape in euclidean geometry. It is the set of all points in a plane that are at a given distance from a given point, the center; equivalently it is the curve traced out by a point that moves so that its distance from a given point is constant.

there are six sides i.e., AB, BC, CD, DE, EF and FA are equal and there are six angles i.e., $\angle ABC$, $\angle BCD$, $\angle CDE$, $\angle DEF$, $\angle EFA$ and $\angle FAB$ are equal.



Therefore, an irregular hexagon is an irregular polygon.

Concave Polygons

A simple polygon that is not convex is called concave, non-convex or reentrant. A concave polygon will always have at least one reflex interior angle—that is, an angle with a measure that is between 180 degrees and 360 degrees exclusive.

Some lines containing interior points of a concave polygon intersect its boundary at more than two points. Some diagonals of a concave polygon lie partly or wholly outside the polygon. Some sidelines of a concave polygon fail to divide the plane into two half-planes one of which entirely contains the polygon. None of these three statements holds for a convex polygon.

As with any simple polygon, the sum of the internal angles of a concave polygon is $\pi (n - 2)$ radians, equivalently $180^\circ \times (n - 2)$, where n is the number of sides. It is always possible to partition a concave polygon into a set of convex polygons. A polynomial-time algorithm for finding a decomposition into as few convex polygons as possible is described by Chazelle & Dobkin (1985).

A triangle can never be concave, but there exist concave polygons with n sides for any $n > 3$. An example of a concave quadrilateral is the dart.

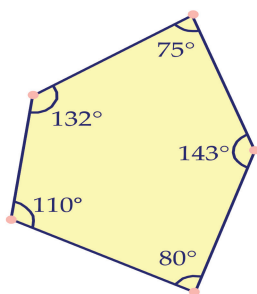
At least one interior angle does not contain all other vertices in its edges and interior.

The convex hull of the concave polygon's vertices, and that of its edges, contains points that are exterior to the polygon.

Convex Polygons

A convex polygon is defined as a polygon with all its interior angles less than 180° . This means that all the vertices of the polygon will point outwards, away from the interior of the shape. Think of it as a 'bulging' polygon. Note that a triangle (3-gon) is always convex.

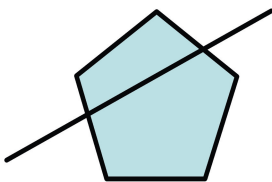
Convex polygon



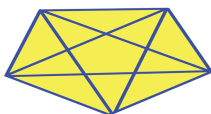
A convex polygon is the opposite of a concave polygon.

In the figure above, drag any of the vertices around with the mouse. Take note of what it takes to make the polygon either convex or concave. Also change the number of sides.

Properties of a Convex Polygon



A line drawn through a convex polygon will intersect the polygon exactly twice, as can be seen from the figure on the left. You can also see that the line will divide the polygon into exactly two pieces.



All the diagonals of a convex polygon lie entirely inside the polygon. See figure on the left. (In a concave polygon, some diagonals will lie outside the polygon).

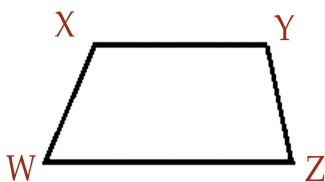
The area of an irregular convex polygon can be found by dividing it into triangles and summing the triangle's areas. See [Area of an Irregular Polygon](#)

Regular Polygons are always convex by definition. See [Regular Polygon Definition](#). In the figure at the top of the page, click on «make regular» to force the polygon to always be a regular polygon. You will see then that, no matter what you do, it will remain convex.

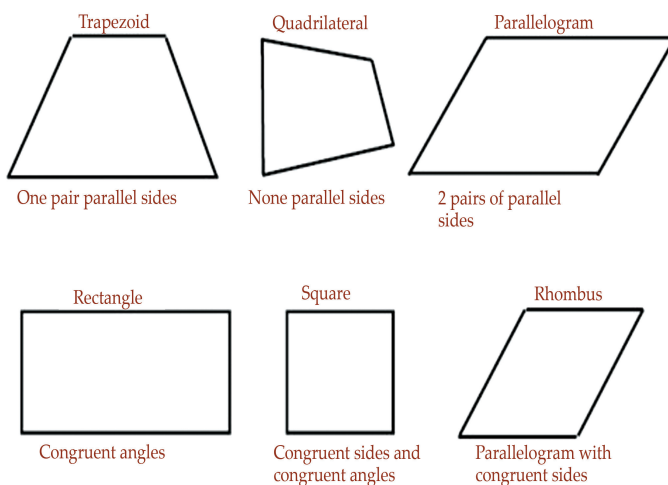
Quadrilateral Polygons

A quadrilateral is a four-sided polygon with four angles. There are many kinds of quadrilaterals. The five most common types are the parallelogram, the rectangle, the square, the trapezoid, and the rhombus.

A quadrilateral is formed by four line segments that intersect at their endpoints. This quadrilateral is named by its endpoints, WXYZ, like a triangle. Squares and rectangles are quadrilaterals that have four right angles.

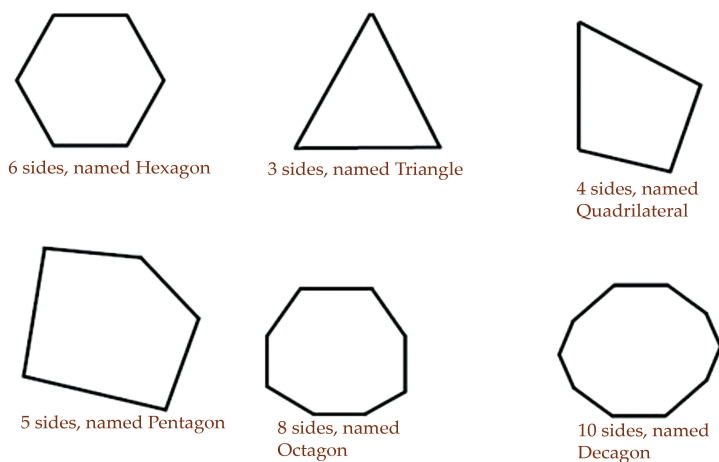


The sum of the angles of a quadrilateral is always 360° . Quadrilaterals can look very different:

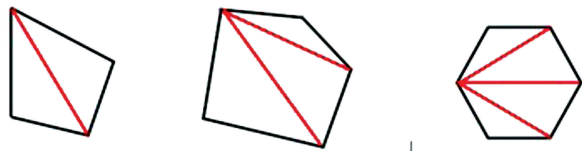


If two figures are similar to each other the corresponding angles are congruent and the corresponding sides are proportional.

Polygons are figures that are formed by three or more line segments. These line segments, even called sides, meet at their endpoints to form vertices.



If we compare the number of sides, diagonals and triangles of a quadrilateral a pentagon and a hexagon we can find a system between the different polygons.



Number of:	Quadrilateral	Pentagon	Hegagon
Sides	4	5	6
Diagonals	1	2	3
Triangles	2	3	4

This can then help us find the sum of the measures of the angles in any polygon. If we compare the number of sides with the number of triangles, we see that there are always 2 less triangles than the number of sides. So, if a polygon has x sides, then the same polygon can form $x - 2$ triangles. The sum of the angles of a polygon is always $180 \cdot (x - 2)$

Pentagon Polygons

A pentagon is any five-sided polygon or 5-gon. The sum of the internal angles in a simple pentagon is 540° .

Properties of pentagons

Interior angle	108°	Like any regular polygon, to find the interior angle we use the formula $(180n - 360)/n$. For a pentagon, $n=5$. See Interior Angles of a Polygon
Exterior Angle	72°	To find the exterior angle of a regular pentagon, we use the fact that the exterior angle forms a linear pair with the interior angle, so in general it is given by the formula $180 - \text{interior angle}$. See Exterior Angles of a Polygon
Area	$1.72 S^2$ approx	Where S is the length of a side. To find the exact area of a regular pentagon or any regular polygon, using various methods, see Area of a Regular Polygon and Area of an Irregular Polygon

Properties of all pentagons

Number of diagonals	5	The number of distinct diagonals possible from all vertices. (In general $\frac{1}{2}n(n-3)$). In the figure above, click on "show diagonals" to see them. See Diagonals of a Polygon
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Number of triangles	3	The number of triangles created by drawing the diagonals from a given vertex. (In general $n-2$). In the figure above, click on “show triangles” to see them. See Triangles of a Polygon
Sum of interior angles	540°	In general $180(n-2)$ degrees . See Interior Angles of a Polygon

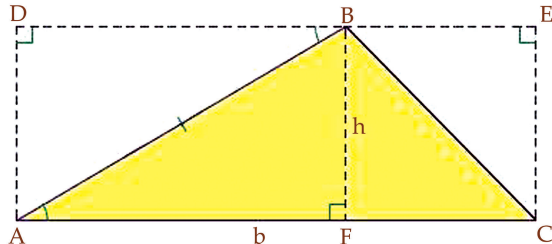
Area and Centroid

Centroid is the center of mass of a two-dimensional figure or three-dimensional solid. Thus the centroid of a two-dimensional figure represents the point at which it could be balanced if it were cut out of, for example, sheet metal. The centroid of a circle or sphere is its center. More generally, the centroid represents the point designated by the mean (see mean, median, and mode) of the coordinates of all the points in a set. If the boundary is irregular, finding the mean requires using calculus (the most general formula for the centroid involves an integral).

Areas

Just as a segment can be measured by comparing it with a unit segment, the area of a polygon or other plane figure can be measured by comparing it with a unit square. The common formulas for calculating areas reduce this kind of measurement to the measurement of certain suitable lengths. The simplest case is a rectangle with sides a and b , which has area ab . By putting a triangle into an appropriate rectangle, one can show that the area of the triangle is half the product of the length of one of its bases and its corresponding height— $bh/2$. One can then compute the area of a general polygon by dissecting it into triangular regions. If a triangle (or more general figure) has area A , a similar triangle (or figure) with a scaling factor of s will have an area of s^2A .

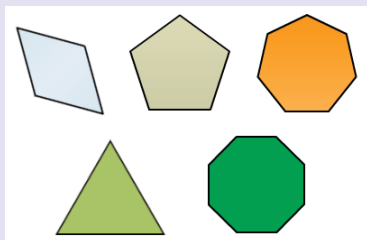
Proof that the area of a triangle = $\frac{1}{2}$ base • height



The right triangle $\triangle AFB$ is $\frac{1}{2}$ of the rectangle $\square ADBF$

Similarly, $\triangle BFC$ is $\frac{1}{2}$ of $\square BECF$.

Thus, the area of $\triangle ABC = \frac{1}{2}$ area of $\square ADEC = \frac{1}{2}AC \cdot BF = \frac{1}{2}bh$. ■



CORE CONCEPT: ANGLES OF POLYGONS

The formula for calculating the sum of the interior angles of a regular polygon is: $(n - 2) \times 180^\circ$ where n is the number of sides of the polygon. This formula comes from dividing the polygon up into triangles using full diagonals. We already know that the interior

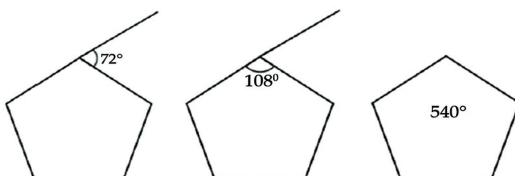
angles of a triangle add up to 180° .

All polygons have both internal angles and external angles. The polygon's external angle is the angle of one side extended and the next side. The external angles of a polygon will always add up to 360° .

You can calculate the internal angles of a polygon using either of the methods below:

Method 1: Assume the polygon is regular.

Begin by working out the size of each external angle. Then use the angles on a straight line rule to work out each angle. Finish up by adding up all the total angles.

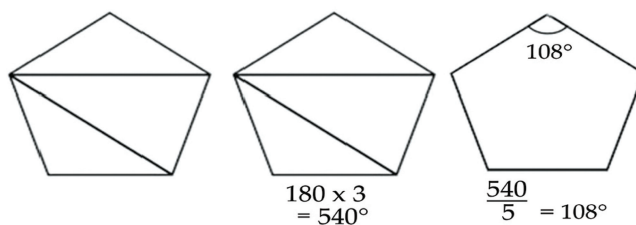


In the example above we are using a pentagon. We therefore divide 360 by 5 to give us an external angle of 72° . If we take 72 from 180 we get 108° . We times this by 5 to give us 540° .

Method 2: Divide the polygon into triangles.

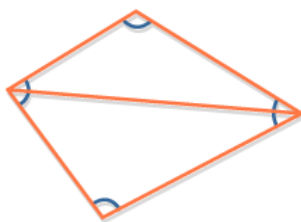
Once you have divided the polygon into triangles you can use the angle sum of a triangle rule to work out the sum of all of the angles. If the shape is regular you can then calculate the size of each angle by dividing the sum of all angles by the number of sides on the shape.

In the example above we just times 180 by 3 (as we can fit 3 triangles into the pentagon). This also gives us 540° . You then then divide this by 5 to give us 108° this is therefore the size of each internal angle.



Angle Properties of Polygons

You might be asked to find angles of polygons. The formula for calculating the sum of the interior angles of a regular polygon is: $(n - 2) \times 180^\circ$ where n is the number of sides of the polygon. This formula comes from dividing the polygon up into triangles using full diagonals.



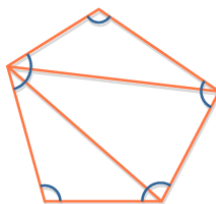
We already know that the interior angles of a triangle add up to 180° . For any polygon, count up how many triangles it can be split into. Then multiply the number of triangles by 180.

This quadrilateral has been divided into two triangles, so the interior angles add up to $2 \times 180 = 360^\circ$.

This pentagon has been divided into three triangles, so the interior angles add up to $3 \times 180 = 540^\circ$.

In the same way, a hexagon can be divided into 4 triangles, a 7-sided polygon into 5 triangles etc.

Can you see the pattern forming? The number of triangles is equal to the number of sides minus 2.



Classification

Polygons are primarily classified by the number of sides.

Convexity and non-convexity

Polygons may be characterized by their convexity or type of non-convexity:

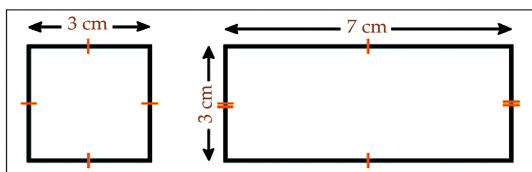
- *Convex*: any line drawn through the polygon (and not tangent to an edge or corner) meets its boundary exactly twice. As a consequence, all its interior angles are less than 180° . Equivalently, any line segment with endpoints on the boundary passes through only interior points between its endpoints.
- *Non-convex*: a line may be found which meets its boundary more than twice. Equivalently, there exists a line segment between two boundary points that passes outside the polygon.
- *Simple*: the boundary of the polygon does not cross itself. All convex polygons are simple.
- *Concave*. Non-convex and simple. There is at least one interior angle greater than 180° .
- *Star-shaped*: the whole interior is visible from at least one point, without crossing any edge. The polygon must be simple, and may be convex or concave.
- *Self-intersecting*: the boundary of the polygon crosses itself. Branko Grünbaum calls these coptic, though this term does not seem to be widely used. The term *complex* is sometimes used in contrast to *simple*, but this usage risks confusion with the idea of a *complex polygon* as one which exists in the complex Hilbert plane consisting of two complex dimensions.
- *Star polygon*: a polygon which self-intersects in a regular way. A polygon cannot be both a star and star-shaped.

A star polygon is a type of non-convex polygon. Only the regular star polygons have been studied in any depth; star polygons in general appear not to have been formally defined.

The Length of the Sides

As well as the number of sides and the angles between sides, the length of each side of shapes is also important.

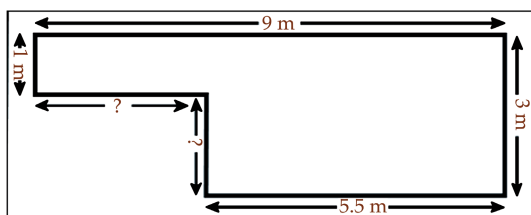
The length of the sides of a plane shape enables you to calculate the shape's perimeter (the distance around the outside of the shape) and area (the amount of space inside the shape).



If your shape is a regular polygon (such as a square in the example above) then it is only necessary to measure one side as, by definition, the other sides of a regular

polygon are the same length. It is common to use tick marks to show that all sides are an equal length.

In the example of the rectangle we needed to measure two sides - the two unmeasured sides are equal to the two measured sides. It is common for some dimensions not to be shown for more complex shapes. In such cases missing dimensions can be calculated.



In the example above, two lengths are missing.

The missing horizontal length can be calculated. Take the shorter horizontal known length from the longer horizontal known length.

$$9\text{m} - 5.5\text{m} = 3.5\text{m}.$$

The same principle can be used to work out the missing vertical length. That is:

$$3\text{m} - 1\text{m} = 2\text{m}.$$

Bringing All the Information Together: Calculating the Area of Polygons

The simplest and most basic polygon for the purposes of calculating area is the quadrangle. To obtain the area, you simply multiple length by vertical height.

For rhombuses, note that vertical height is not the length of the sloping side, but the vertical distance between the two horizontal lines.

This is because a rhombus is essentially a rectangle with a triangle cut off one end and pasted onto the other:

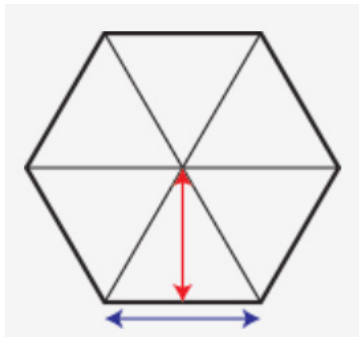
You can see that if you remove the left hand blue triangle, and stick it onto the other end, the rectangle becomes a rhombus.



The area is length (the top horizontal line) multiplied by height, the vertical distance between the two horizontal lines.

To work out the area of a triangle, you multiple length by vertical height (that is, the vertical height from the bottom line to the top point), and halve it. This is essentially because a triangle is half a rectangle.

To calculate the area of any regular polygon, the easiest way is to divide it into triangles, and use the formula for the area of a triangle.



So, for a hexagon, for example:

You can see from the diagram that there are six triangles.

The area is:

Height (red line) $\times$ length of side (blue line) $\times 0.5 \times 6$
(because there are six triangles).

You can also work out the area of any regular polygon using trigonometry, but that's rather more complicated.

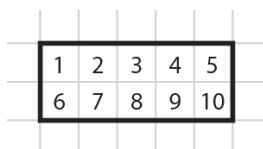
Calculating Area

Area is a measure of how much space there is inside a shape. Calculating the area of a shape or surface can be useful in everyday life – *for example* you may need to know how much paint to buy to cover a wall or how much grass seed you need to sow a lawn.

A concave polygon has at least one interior angle greater than 180 degrees, and there's at least one line segment that goes outside the polygon.

Calculating Area Using the Grid Method

When a shape is drawn on a scaled grid you can find the area by counting the number of grid squares inside the shape.



In order to find an area value, using the grid method, we need to know the size that a grid square represents.

This example uses centimeters but the same method applies for any unit of **length** or distance. You could, for example be using inches, meters, miles, feet etc.

In this example each grid square has a width of 1cm and a height of 1cm. In other words each grid square is one 'square centimeter'. Count the grid squares inside the large square to find its area. There are 16 small squares so the area of the large

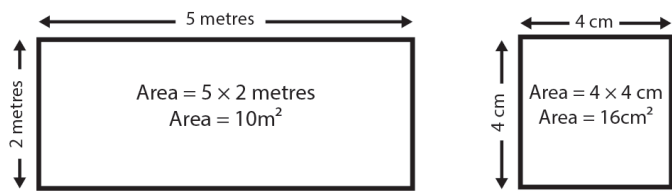
Length is the most extended dimension of an object, in the international system of quantities, length is any quantity with dimension distance.

square is 16 square centimeters. In mathematics we abbreviate ‘square centimeters’ to cm^2 . The 2 means ‘squared’.

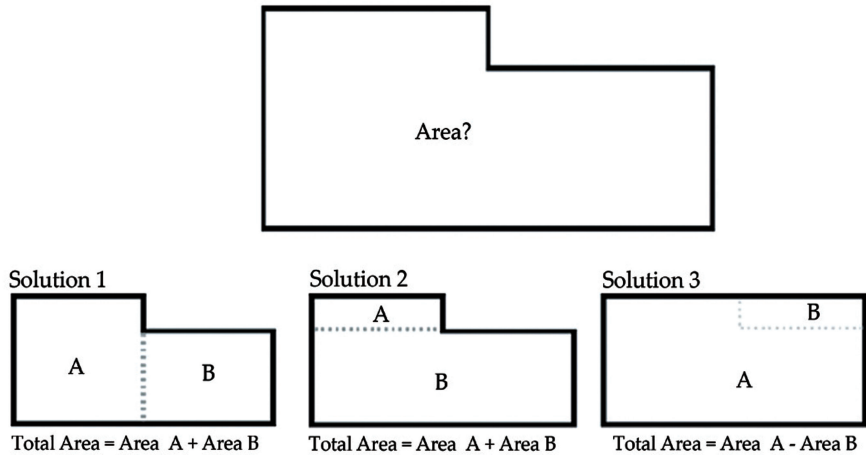
Areas of Squares and Rectangles (and parallelograms)

The simplest (and most commonly used) area calculations are for squares and rectangles. To find the area of a rectangle multiply its height by its width. For a square you only need to find the length of one of the sides (as each side is the same length) and then multiply this by itself to find the area. This is the same as saying length^2 or length squared.

It is good practice to check that a shape is actually a square by measuring two sides. For example the wall of a room may look like a square but when you measure it you find it is actually a rectangle.



Often, in real life, shapes can be more complex. *For example*, imagine you want to find the area of a floor, so that you can order the right amount of carpet. A typical floor-plan of a room may not consist of a simple rectangle or square:



In this example, and other examples like it, the trick is to split the shape into several rectangles (or squares). It doesn't matter how you split the shape - any of the three solutions will result in the same answer.

Solution 1 and 2 require that you make two shapes and add their areas together to find the total area.

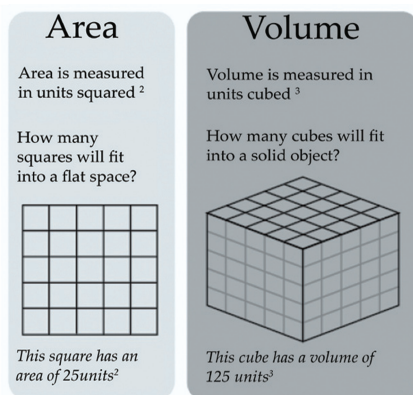
For solution 3 you make a larger shape (A) and subtract the smaller shape (B) from it to find the area.

Calculating Volume

Volume is the measure of how much space there is within a three-dimensional object. Our page on three-dimensional shapes explains the basics of such shapes.

In the real world, calculating volume is probably not something that you will use as often as calculating area.

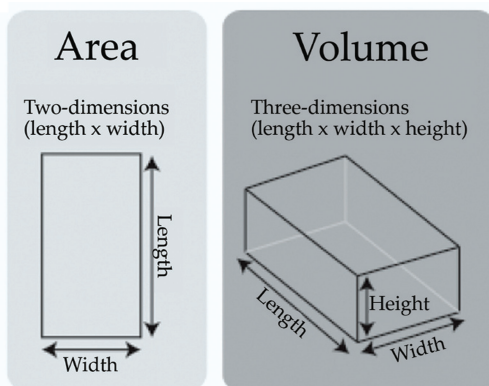
However it can still be important. Being able to calculate volume will enable you to, *for example*, work out how much packing space you have when moving house, how much office space you need, or how much jam you can fit into a jar. It can also be useful for understanding what the media mean when they talk about the capacity of a dam or the flow of a river.



Basic Formulas for Calculating Volume

Volume of Rectangle-Based Solids

Whereas the basic formula for the area of a rectangular shape is $\text{length} \times \text{width}$, the basic formula for volume is $\text{length} \times \text{width} \times \text{height}$.



How you refer to the different dimensions does not change the calculation: you may, *for example*, use 'depth' instead of 'height'. The important thing is that the three dimensions are multiplied together. You can multiply in which ever order you like as it would not change the answer

Volume is the quantity of three-dimensional space enclosed by a closed surface, *for example*, the space that a substance (solid, liquid, gas, or plasma) or shape occupies or contains.

A box with the dimensions 15cm width, 25cm length and 5 cm height has a volume of:

$$15 \times 25 \times 5 = 1875\text{cm}^3$$

Volume of Prisms and Cylinders

This basic formula can be extended to cover the **volume** of cylinders and prisms too. Instead of a rectangular end, you simply have another shape: a circle for cylinders, a triangle, hexagon or, indeed, any other polygon for a prism.

Effectively, for cylinders and prisms, the volume is the area of one side multiplied by the depth or height of the shape.

The basic formula for volume of prisms and cylinders is therefore:

Area of the end shape $\times$ the height/depth of the prism/cylinder.

Volume of Cones and Pyramids

The same principle as above (width $\times$ length $\times$ height) holds for calculating the volume of a cone or a pyramid except that, because they come to a point, the volume is only a proportion of the total that it would be if they continued in the same shape right through.

Volume of Cones and Pyramids

The same principle as above (width $\times$ length $\times$ height) holds for calculating the volume of a cone or a pyramid except that, because they come to a point, the volume is only a proportion of the total that it would be if they continued in the same shape right through.

Area of the base or end shape $\times$ the height of the cone/pyramid $\times \frac{1}{3}$

Radius of a circle or sphere is any of the line segments from its center to its perimeter, and in more modern usage, it is also the length of any of them.

Refer back to our page calculating area if you cannot remember how to calculate the area of a circle or triangle.

For example, to calculate the volume of a cone with a **radius** of 5cm and a height of 10cm:

The area within a circle = πr^2 (where π (pi) is approximately 3.14 and r is the radius of the circle). In this example, area of base (circle) = $\pi r^2 = 3.14 \times 5 \times 5 = 78.5\text{cm}^2$.

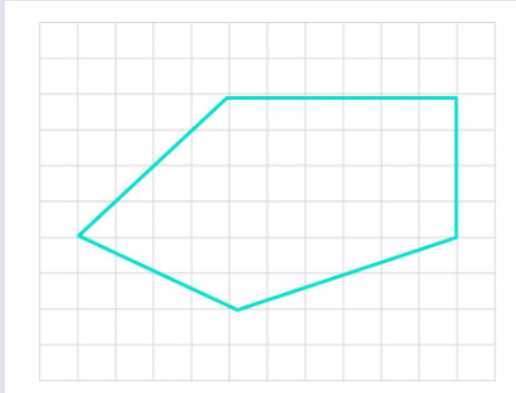
$$78.5 \times 10 = 785$$

$$785 \times \frac{1}{3} = 261.6667\text{cm}^3$$

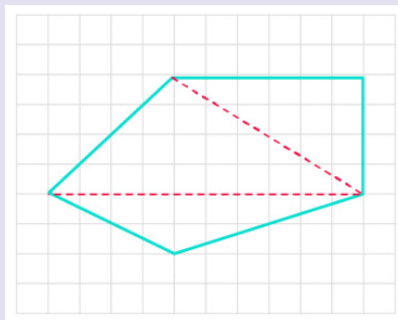
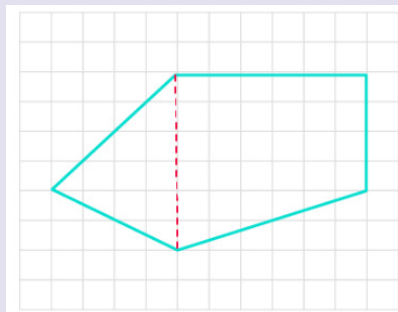
CONCEPT: HOW TO FIND AREA OF POLYGON WITH N-SIDES?

For finding the area of a polygon which is not regular or its formula is not defined, we split the figure into triangles, squares, trapezium, etc. The purpose is to visualize the given geometry as a combination of geometries for which we know how to calculate the area. We then calculate the area for each of the part and then add them up to obtain the area of the polygon.

For example, consider the polygon shown below:



This polygon can be divided into a combination of triangles and trapezium.



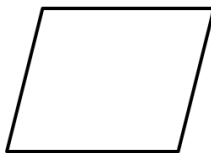
We can calculate the area considering any of the above divisions. In the first figure, we can sum up the area of the triangle as well as the trapezium to obtain the area of the polygon. In the second one, we add the areas of the three triangles to get the area of a polygon. In either case, the result is the same.

SUMMARY

- Polygon is a fundamental geometric concept that plays a significant role in mathematics and various fields of science and engineering. A polygon is a two-dimensional closed shape consisting of straight sides and vertices. These vertices are the points where the sides of the polygon intersect.
- A polygon is a 2D shape which is made up of straight line segments. A regular polygon has all sides and angles equal (e.g. a square is a regular quadrilateral). A triangle with all sides and angles equal is known as an equilateral triangle.
- A closed plane figure bounded by three or more line segments is called a polygon. The polygon is a Greek word in which the meaning of ‘poly’ is ‘many’ and the meaning of the ‘gon’ is ‘angle’.
- A polygon is a plane shape that has at least three sides. Etymologically, they have multiple (poly) angles (gonia). In other words, a two-dimensional object which is surrounded by finite number of straight lines binding in the form of a closed loop, is categorized as a polygon.
- An irregular polygon is a polygon that is not regular. That is, an irregular polygon is a polygon that does not have all sides of equal lengths and all interior angles equal.
- A simple polygon that is not convex is called concave, non-convex or reentrant. A concave polygon will always have at least one reflex interior angle—that is, an angle with a measure that is between 180 degrees and 360 degrees exclusive.
- A convex polygon is defined as a polygon with all its interior angles less than 180° . This means that all the vertices of the polygon will point outwards, away from the interior of the shape.
- A quadrilateral is formed by four line segments that intersect at their endpoints. This quadrilateral is named by its endpoints, WXYZ, like a triangle. Squares and rectangles are quadrilaterals that have four right angles.

MULTIPLE CHOICE QUESTIONS

1. What is the circle which connects all the vertices of a polygon called?
 - a. Semi-circle
 - b. Incircle
 - c. Outer circle
 - d. Circumcircle
2. What is a polygon with 50 sides called as?
 - a. Pentahehexagon
 - b. Pentagon
 - c. Pentacontagon
 - d. Pentacontagion
3. What will be the sum of exterior angle of a regular polygon?
 - a. 360°
 - b. 180°
 - c. 540°
 - d. 120°
4. What parameter is used to find the area and perimeter of a polygon?
 - a. Curve
 - b. Size
 - c. Length
 - d. Angle
5. What will be the area of the given figure if the length of one side is 10 cm and the height of the parallelogram is 15 cm?



- a. 225 cm^2
- b. 1225 cm^2
- c. 150 cm^2
- d. 75 cm^2

Answer

1. (d) 2. (c) 3. (a) 4. (c) 5. (c)

REVIEW QUESTIONS

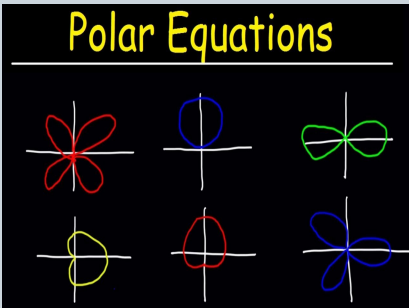
1. What do you understand by polygon parts?
2. Explain the properties of a regular polygon.
3. Define the angle properties of polygons.
4. Explain the types of polygon.
5. How to calculate area and volume? Explain.

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POLAR COORDINATES AND CURVE

Polar Equations



The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

After studying this chapter, you will understand the core concepts of:

- Polar Coordinates
- Polar Curves

INTRODUCTION

Polar coordinates are a two-dimensional coordinate system used to describe the position of a point in a plane using distance and angle. This system is especially useful when dealing with circular or rotational symmetrical phenomena, such as curves and patterns that revolve around a central point.

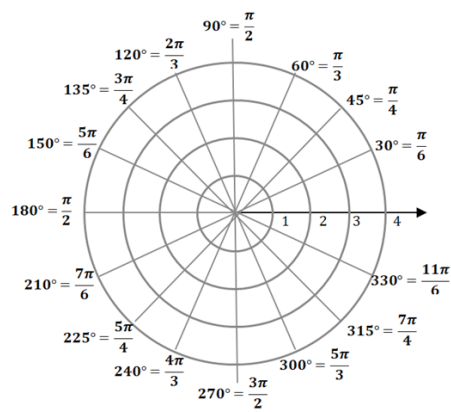
In the polar coordinate system, a point is defined by two values: the radial distance (r) and the angular direction (θ). The radial distance represents the distance between the point and the origin (the central point), and the angular direction represents the angle formed between a reference axis (usually the positive x -axis) and the line connecting the origin and the point.

The conversion between polar coordinates (r, θ) and Cartesian coordinates (x, y) is as follows:

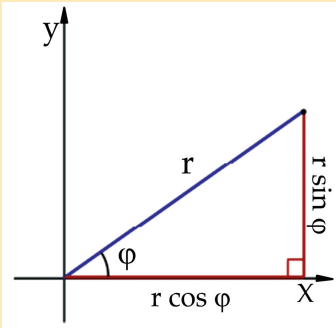
- $x = r * \cos(\theta)$
- $y = r * \sin(\theta)$

Here, r is a non-negative real number, and θ is typically measured in radians. The angle θ can be positive or negative, with positive values measured counterclockwise from the reference axis, and negative values measured clockwise.

A polar curve is simply the resulting graph of a polar equation defined by r and θ . We use polar grids or polar planes to plot the polar curve and this graph is defined by all sets of (r, θ) , that satisfy the given polar equation, $r = f(\theta)$.



The graph above is a standard example of a polar grid. We call the center the pole and the horizontal axis extending to the right is called the polar axis. The angles outside are the angles that return exact values and this is why they are great references for polar grids as well.



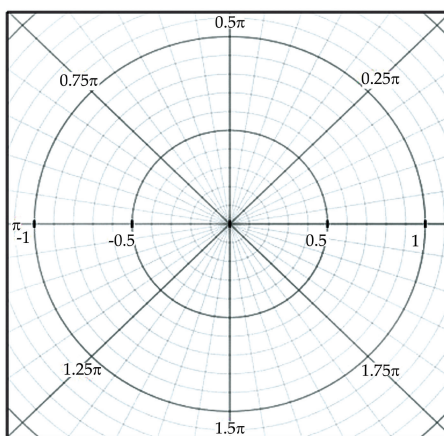
CORE CONCEPT: POLAR COORDINATES

The polar coordinate system is a two-dimensional coordinate system in which each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

Polar coordinates provide a method of rendering graphs and indicating the positions of points on a two-dimensional (2D) surface. The polar coordinate system is employed in mathematics, physics, engineering, navigation, robotics, and other sciences.

In polar coordinate system, a point is specified by its angle to a ray and distance from the Pole.

In polar coordinate system, the plane is marked a point called Pole (sometimes also called Origin or Center). A horizontal ray is drawn from this point to the right; we call this the reference ray. A point in the plane is specified by two numbers $\{r, \theta\}$. The θ is the angle from the reference ray. (positive angles have counter-clockwise sweep) The r is the distance from the point to the Pole. If r is negative, then the point is on the opposite side with respect to the Pole.



Rectangular coordinates (those using x-axis and y-axis) are very inconvenient. In rectangular coordinates, we describe points as being a certain distance along the x-axis and a certain distance along the y-axis.

Rectangular coordinate is also known as the Cartesian coordinate system after Rene Descartes, who popularized its use in analytic geometry.

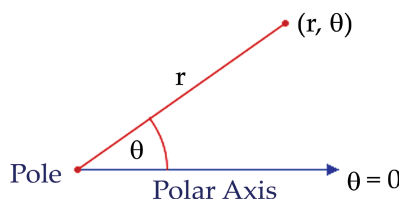
But certain functions are very complicated if we use the rectangular coordinate system. Such functions may be much simpler in the polar coordinate system, which allows us to describe and graph certain functions in a very convenient way.

Polar coordinates work in much the same way that we have seen in trigonometry (radians and arc length, where we used r and θ) and in the polar form of complex numbers (where we also saw r and θ).

In polar coordinates, we describe points as being a certain distance (r) from the pole (the origin) and at a certain angle (θ) from the positive horizontal axis (called the polar axis).

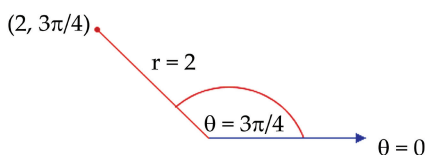
The coordinates of a point in polar coordinates are written as (r, θ)

The graph of the point (r, θ) is as follows:



Example:

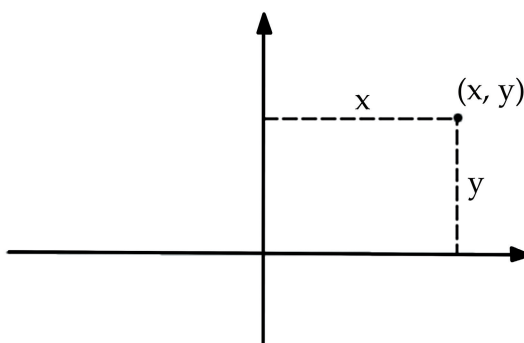
The point described in polar coordinates by $(2, 3\pi/4)$ would look like this:



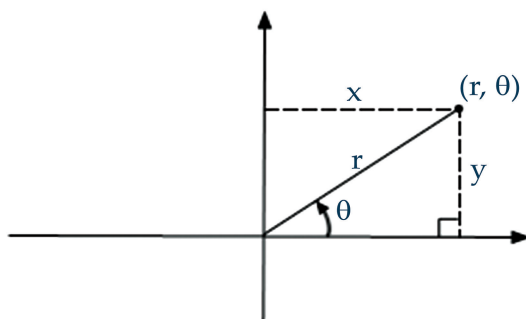
We use polar graph paper for drawing points in polar coordinates.

NOTE: Angles can be in degrees or radians for polar coordinates.

Coordinate systems are really nothing more than a way to define a point in space. For instance in the Cartesian coordinate system a point is given the coordinates (x, y) and we use this to define the point by starting at the origin and then moving x units horizontally followed by y units vertically. This is shown in the sketch below.



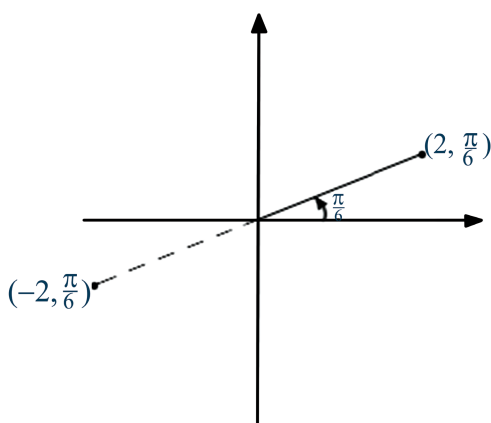
This is not, however, the only way to define a point in two dimensional spaces. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. This is shown in the sketch below.



Coordinates in this form are called polar coordinates.

The above discussion may lead one to think that r must be a positive number.

However, we also allow r to be negative. Below is a sketch of the two points $(2, \frac{\pi}{6})$ and $(-2, \frac{\pi}{6})$.



From this sketch we can see that if r is positive the point will be in the same quadrant as θ . On the other hand if r is negative the point will end up in the quadrant exactly opposite θ . Notice as

well that the coordinates $(-2, \frac{\pi}{6})$ describe the same point as the coordinates $(2, \frac{7\pi}{6})$ do. The coordinates $(2, \frac{7\pi}{6})$ tells us to rotate an angle of $\frac{7\pi}{6}$ from the positive x-axis, this would put us on the dashed line in the sketch above, and then move out a distance of 2.

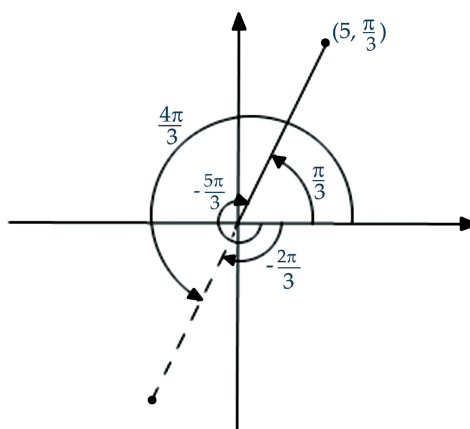
In planar geometry, an angle is the figure formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle.

Cartesian coordinate system is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length.

This leads to an important difference between **Cartesian coordinates** and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this is not true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$(5, \frac{\pi}{3}) = (5, -\frac{5\pi}{3}) = (-5, \frac{4\pi}{3}) = (-5, -\frac{2\pi}{3})$$

Here is a sketch of the angles used in these four sets of coordinates.



In the second coordinate pair we rotated in a clock-wise direction to get to the point. We should not forget about rotating in the clock-wise direction. Sometimes it's what we have to do.

The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative r to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.

These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotation about the axis system as we want then there are an infinite number of coordinates for the same point. In fact the point (r, θ) can be represented by any of the following coordinate pairs.

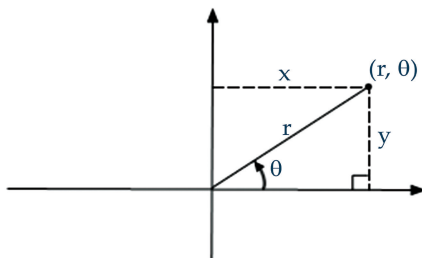
$$(r, \theta + 2\pi n)$$

$$(-r, \theta + (2n + 1)\pi)$$

where n is any integer.

Next we should talk about the origin of the coordinate system. In polar coordinates the origin is often called the pole. Because we are not actually moving away from the origin/pole we know that $r = 0$. However, we can still rotate around the system by any angle we want and so the coordinates of the origin/pole are $(0, \theta)$.

Now that we have got a grasp on polar coordinates we need to think about converting between the two coordinate systems. We'll start out with the following sketch reminding us how both coordinate systems work.



Polar to Cartesian Conversion Formulas

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Converting from Cartesian is almost as easy. Let us first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r \cos \theta)^2 + (r \sin \theta)^2 \\ &= r^2 \cos^2 \theta + r^2 \sin^2 \theta \\ &= r^2 (\cos^2 \theta + \sin^2 \theta) = r^2 \end{aligned}$$

This is a very useful formula that we should remember, however we are after an equation for r so let us take the square root of both sides.

This gives,

$$r = \sqrt{x^2 + y^2}$$

Note that technically we should have a plus or minus in front of the root since we know that r can be either positive or negative. We will run with the convention of positive r here.

Getting an equation for θ is almost as simple. We will start with,

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

Taking the inverse tangent of both sides gives,

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

We will need to be careful with this because inverse tangents only return values in

the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \Pi$.

Summarizing then gives the following formulas for converting from Cartesian coordinates to polar coordinates.

Cartesian to Polar Conversion Formulas

$$r^2 = x^2 + y^2 \qquad r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

Common Polar Coordinate Graphs

Let us identify a few of the more common graphs in polar coordinates. We will also take a look at a couple of special polar graphs.

Lines

Some lines have fairly simple equations in polar coordinates.

- $\theta = \beta$. We can see that this is a line by converting to Cartesian coordinates as follows

$$\theta = \beta$$

$$\tan^{-1} \left(\frac{y}{x} \right) = \beta$$

$$\frac{y}{x} = \tan \beta$$

$$y = (\tan \beta) x$$

This is a line that goes through the origin and makes an angle of β with the positive x -axis. Or, in other words it is a line through the origin with slope of $\tan \beta$.

- $r \cos \theta = a$. This is easy enough to convert to Cartesian coordinates to $x = a$. So, this is a vertical line.
- $r \sin \theta = b$. Likewise, this converts to $y = b$ and so is a horizontal line.

Circles

Let us take a look at the equations of circles in polar coordinates.

- $r = A$. This equation is saying that no matter what angle we have got the distance from the origin must be a . If you think about it that is exactly the definition of a circle of radius a centered at the origin.
- So, this is a circle of radius a centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the

corresponding equation in Cartesian coordinates.

- $r=2a\cos \theta$. We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

This is a circle of radius $|a|$ and center $(a, 0)$. Note that a might be negative (as it was in our example above) and so the absolute value bars are required on the **radius**. They should not be used however on the center.

Radius of a circle or sphere is any of the line segments from its center to its perimeter, and in more modern usage, it is also their length.

- $r=2b\sin \theta$

This is similar to the previous one. It is a circle of radius $|b|$ and center $(0,b)$.

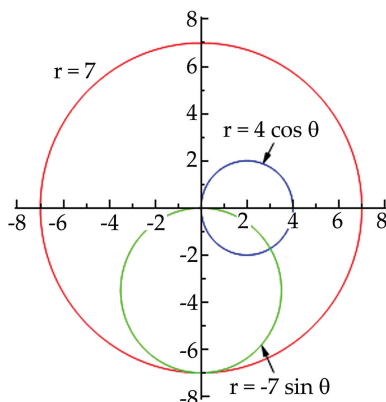
- $r=2a\cos \theta+2b\sin \theta$. This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2+b^2}$

and center (a, b) . In other words, this is the general equation of a circle that is not centered at the origin.

Example: Graph $r=7$, $r=4\cos \theta$; and $r=-7\sin \theta$ on the same axis system.

Solution: The first one is a circle of radius 7 centered at the origin. The second is a circle of radius 2 centered at $(2,0)$. The third is a circle of

radius $\frac{7}{2}$ centered at $\left(0, -\frac{7}{2}\right)$. Here is the graph of the three equations.



Note that it takes a range of $0 \leq \theta \leq 2\pi$ for a complete graph of $r=a$ and it only takes a range of $0 \leq \theta \leq \pi$ to graph the other circles given here.

Slopes in Polar Coordinates

When we describe a curve using polar coordinates, it is still a curve in the x - y plane. We would like to be able to compute slopes and areas for these curves using polar coordinates. We have seen that $x = r \cos \theta$ and $y = r \sin \theta$ describe the relationship between polar and rectangular coordinates. If in turn we are interested in a curve given by $r = f(\theta)$, then we can write $x = f(\theta) \cos \theta$ and $y = f(\theta) \sin \theta$, describing x and y in terms of θ alone. The first of these equations describes θ implicitly in terms of x , so using the chain rule we may compute

$$\frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx}.$$

Since $d\theta/dx = 1/(dx/d\theta)$, we can instead compute

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta) \cos \theta - f(\theta) \sin \theta}{-f'(\theta) \sin \theta - f(\theta) \cos \theta}.$$

Example : Find the points at which the curve given by $r = 1 + \cos \theta$ has a vertical or horizontal tangent line. Since this function has period 2π , we may restrict our attention to the interval $[0, 2\pi)$ or $(-\pi, \pi]$, as convenience dictates. First, we compute the slope:

$$\frac{dy}{dx} = \frac{(1 + \cos \theta) \cos \theta - \sin \theta \sin \theta}{-(1 + \cos \theta) \sin \theta - \sin \theta \cos \theta} = \frac{\cos \theta + \cos^2 \theta - \sin^2 \theta}{-\sin \theta - 2 \sin \theta \cos \theta}.$$

This fraction is zero when the numerator is zero (and the denominator is not zero). The numerator is $2 \cos^2 \theta + \cos \theta - 1$ so by the quadratic formula

$$\cos \theta = \frac{-1 \pm \sqrt{1 + 4 \cdot 2}}{4} = -1 \quad \text{or} \quad \frac{1}{2}.$$

This means θ is π or $\pm\pi/3$. However, when $\theta = \pi$, the denominator is also 0, so we cannot conclude that the tangent line is horizontal. Setting the denominator to zero we get

$$-\theta - 2 \sin \theta \cos \theta = 0$$

$$\sin \theta (1 + 2 \cos \theta) = 0,$$

so either $\sin \theta = 0$ or $\cos \theta = -1/2$. The first is true when θ is 0 or π , the second when θ is $2\pi/3$ or $4\pi/3$. However, as above, when $\theta = \pi$, the numerator is also 0, so we cannot conclude that the tangent line is vertical. Figure 1 shows points corresponding to θ equal to 0, $\pm\pi/3$, $2\pi/3$ and $4\pi/3$ on the graph of the function. Note that when $\theta = \pi$ the curve hits the origin and does not have a tangent line.

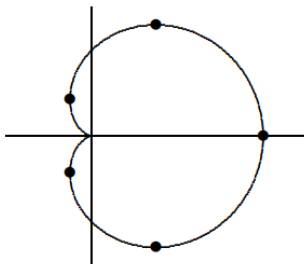


Figure 1: Points of vertical and horizontal tangency for $r = 1 + \cos \theta$.

We know that the second derivative $f''(x)$ is useful in describing functions, namely, in describing concavity. We can compute $f''(x)$ in terms of polar coordinates as well. We already know how to write $dy/dx = y'$ in terms of θ , then

$$\frac{d}{dx} \frac{dy}{dx} = \frac{dy'}{dx} = \frac{dy'}{d\theta} \frac{d\theta}{dx} = \frac{dy'/d\theta}{dx/d\theta}.$$

Example: We find the second derivative for the cardioid $r = 1 + \cos \theta$:

$$\begin{aligned} \frac{d}{d\theta} \frac{\cos \theta + \cos^2 \theta - \sin^2 \theta}{-\sin \theta - 2 \sin \theta \cos \theta} \cdot \frac{1}{dx/d\theta} &= \dots = \frac{3(1 + \cos \theta)}{(\sin \theta + 2 \sin \theta \cos \theta)^2} \cdot \frac{1}{-(\sin \theta + 2 \sin \theta \cos \theta)} \\ &= \frac{-3(1 + \cos \theta)}{(\sin \theta + 2 \sin \theta \cos \theta)^3}. \end{aligned}$$

The ellipsis here represents rather a substantial amount of algebra. We know from above that the cardioid has horizontal tangents at $\pm\pi/3$; substituting these values into the second derivative we get $y''(\pi/3) = -\sqrt{3}/2$ and $y''(-\pi/3) = \sqrt{3}/2$, indicating concave down and concave up respectively. This agrees with the graph of the function.

Areas in Polar Coordinates

We can use the equation of a curve in polar coordinates to compute some areas bounded by such curves. The basic approach is the same as with any application of integration: find an approximation that approaches the true value. For areas in rectangular coordinates, we approximated the region using rectangles; in polar coordinates, we use sectors of circles, as depicted in figure 2. Recall that the area of a sector of a circle is $\alpha r^2/2$, where α is the angle subtended by the sector. If the curve is given by $r = f(\theta)$, and the angle subtended by a small sector is $\Delta\theta$, the area is $(\Delta\theta)(f(\theta))^2/2$. Thus we approximate the total area as

$$\sum_{i=0}^{n-1} \frac{1}{2} f(\theta_i)^2 \Delta\theta.$$

In the limit this becomes

$$\int_a^b \frac{1}{2} f(\theta)^2 d\theta.$$

Example: We find the area inside the cardioid $r = 1 + \cos \theta$.

$$\int_0^{2\pi} \frac{1}{2} (1 + \cos \theta)^2 d\theta = \frac{1}{2} \int_0^{2\pi} 1 + 2 \cos \theta + \cos^2 \theta d\theta = \frac{1}{2} \left(\theta + 2 \sin \theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \Big|_0^{2\pi} = \frac{3\pi}{2}.$$

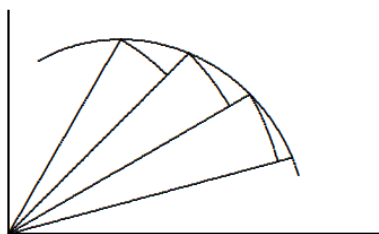


Figure 2: Approximating area by sectors of circles.

Example: We find the area between the circles $r = 2$ and $r = 4 \sin \theta$, as shown in figure 3. The two curves intersect where $2 = 4 \sin \theta$, or $\sin \theta = 1/2$, so $\theta = \pi/6$ or $5\pi/6$. The area we want is then

$$\frac{1}{2} \int_{\pi/6}^{5\pi/6} 16 \sin^2 \theta - 4 d\theta = \frac{4}{3}\pi + 2\sqrt{3}.$$

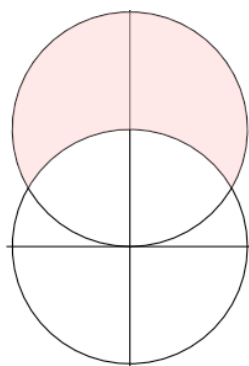


Figure 3: An area between curves.

This example makes the process appear more straightforward than it is. Because points have many different representations in polar coordinates, it is not always so easy to identify points of intersection.

Example: We find the shaded area in the first graph of figure 4 as the difference of the other two shaded areas. The cardioid is $r = 1 + \sin \theta$ and the circle is $r = 3 \sin \theta$. We attempt to find points of intersection:

$$1 + \sin \theta = 3 \sin \theta$$

$$1 = 2 \sin \theta$$

$$1/2 = \sin \theta.$$

This has solutions $\theta = \pi/6$ and $5\pi/6$; $\pi/6$ corresponds to the intersection in the first quadrant that we need. Note that no solution of this equation corresponds to the intersection point at the origin, but fortunately that one is obvious. The cardioid goes

through the origin when $\theta = -\pi/2$; the circle goes through the origin at multiples of π , starting with 0.

Now the larger region has area

$$\frac{1}{2} \int_{-\pi/2}^{\pi/6} (1 + \sin \theta)^2 d\theta = \frac{\pi}{2} - \frac{9}{16}\sqrt{3}$$

and the smaller has area

$$\frac{1}{2} \int_0^{\pi/6} (3 \sin \theta)^2 d\theta = \frac{3\pi}{8} - \frac{9}{16}\sqrt{3}$$

so the area we seek is $\pi/8$.

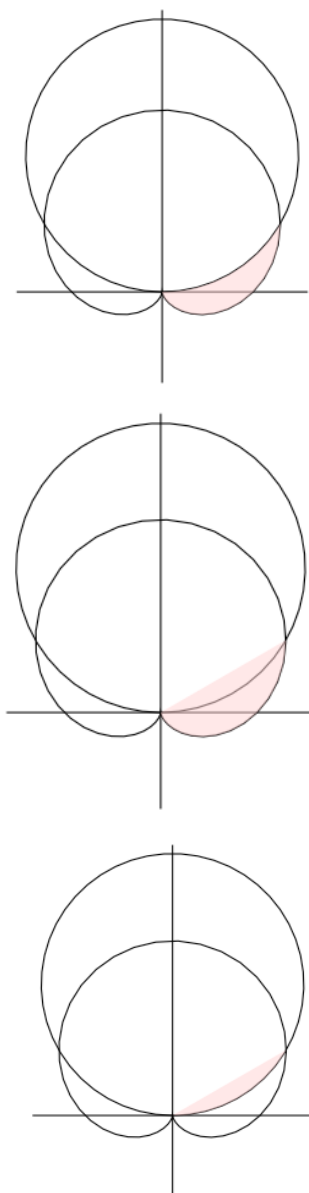


Figure 4: An area between curves.

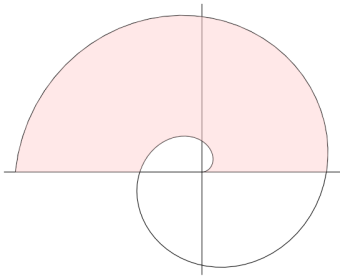
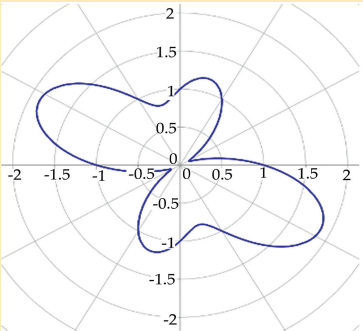


Figure 5: An area bounded by the spiral of Archimedes.



CORE CONCEPT: POLAR CURVES

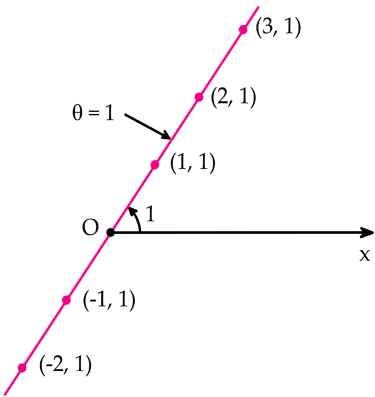
Polar curve is a shape constructed using the polar coordinate system. Polar curves are defined by points that are a variable distance from the origin (the pole) depending on the angle measured off the positive x-axis.

The graph of a polar equation $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

Example: Sketch the polar curve $\theta = 1$.

Solution: This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis. Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

Polar axis is the reference line used in the polar coordinate system for measuring angles. It serves as the equivalent of the x-axis in the Cartesian coordinate system.



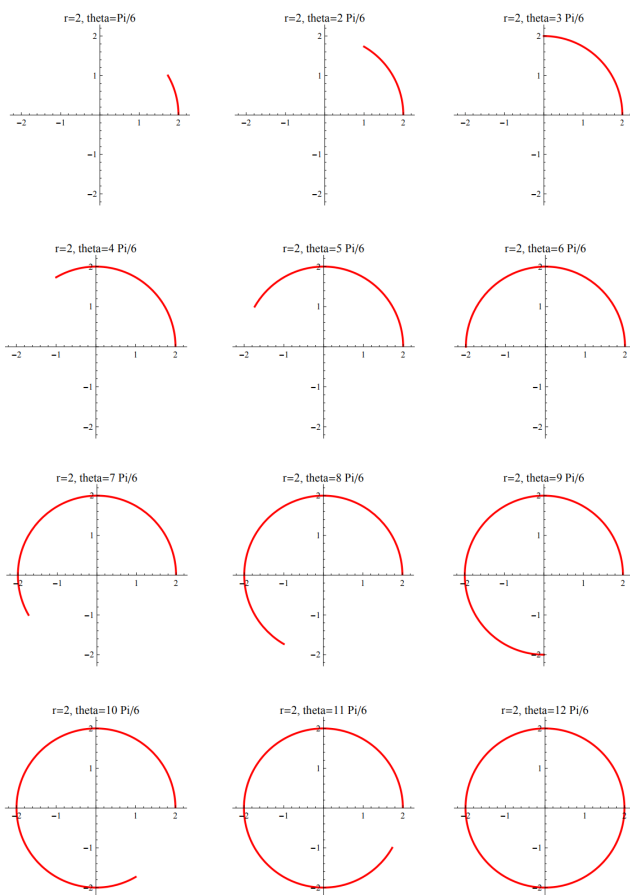
Example: Sketch the following curves:

(a) $r = 2, 0 \leq \theta \leq 2\pi$.

(b) $r = 2 \cos \theta, 0 \leq \theta \leq \pi$.

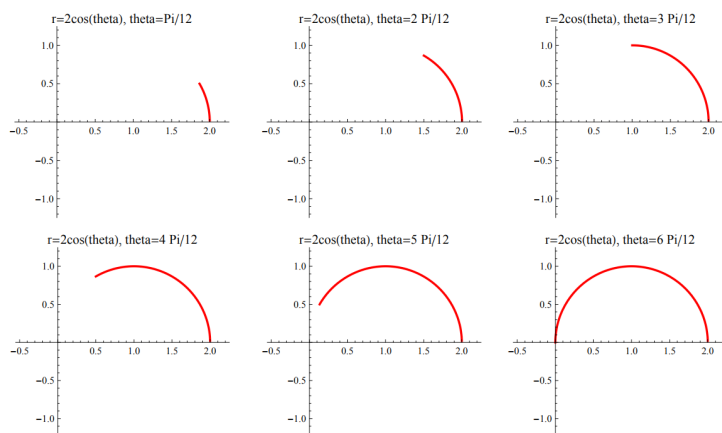
Example: Sketch the curve $r = 2, 0 \leq \theta \leq 2\pi$.

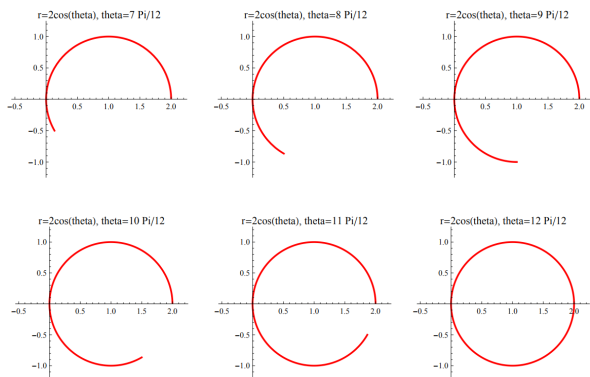
Solution: We have



Example: Sketch the curve $r = 2 \cos \theta, 0 \leq \theta \leq \pi$.

Solution: We have





Example: Express the polar equation $r = 2 \cos \theta$ in rectangular coordinates.

Solution: We use the formulas $r^2 = x^2 + y^2$ and $x = r \cos \theta$. We have

$$r = 2 \cos \theta$$

$$r^2 = 2r \cos \theta$$

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x - 1)^2 + y^2 = 1$$

Example: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5 \sec \theta$

(b) $r = 2 \sin \theta$

(c) $r = 2 + 2 \cos \theta$

Example: Express the polar equation in rectangular coordinates. If possible, determine the graph of the equation from its rectangular form.

(a) $r = 5 \sec \theta$

(b) $r = 2 \sin \theta$

(c) $r = 2 + 2 \cos \theta$

Solution: We use the formulas $r^2 = x^2 + y^2$, $x = r \cos \theta$ and $y = r \sin \theta$.

(a) We have

$$r = 5 \sec \theta$$

$$r \cos \theta = 5$$

$$x = 5$$

(b) We have

$$r = 2 \sin \theta$$

$$r^2 = 2r \sin \theta$$

$$x^2 + y^2 = 2y$$

$$x^2 + y^2 - 2y = 0$$

$$x^2 + y^2 - 2y + 1 = 1$$

$$x^2 + (y - 1)^2 = 1$$

(c) We have

$$r = 2 + 2 \cos \theta$$

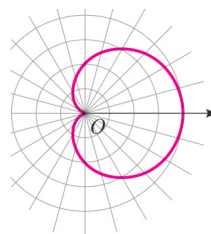
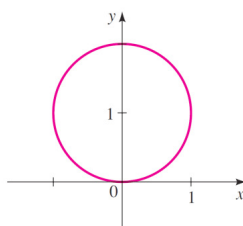
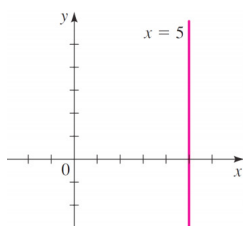
$$r^2 = 2r + 2r \cos \theta$$

$$x^2 + y^2 = 2r + 2x$$

$$x^2 + y^2 - 2x = 2r$$

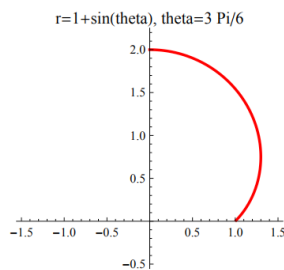
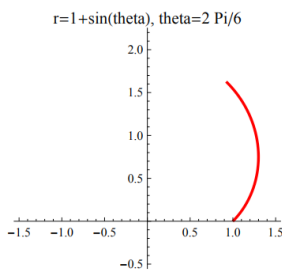
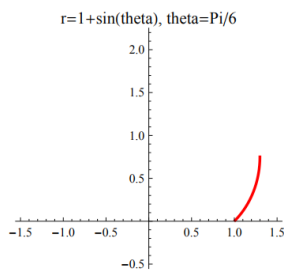
$$(x^2 + y^2 - 2x)^2 = 4r^2$$

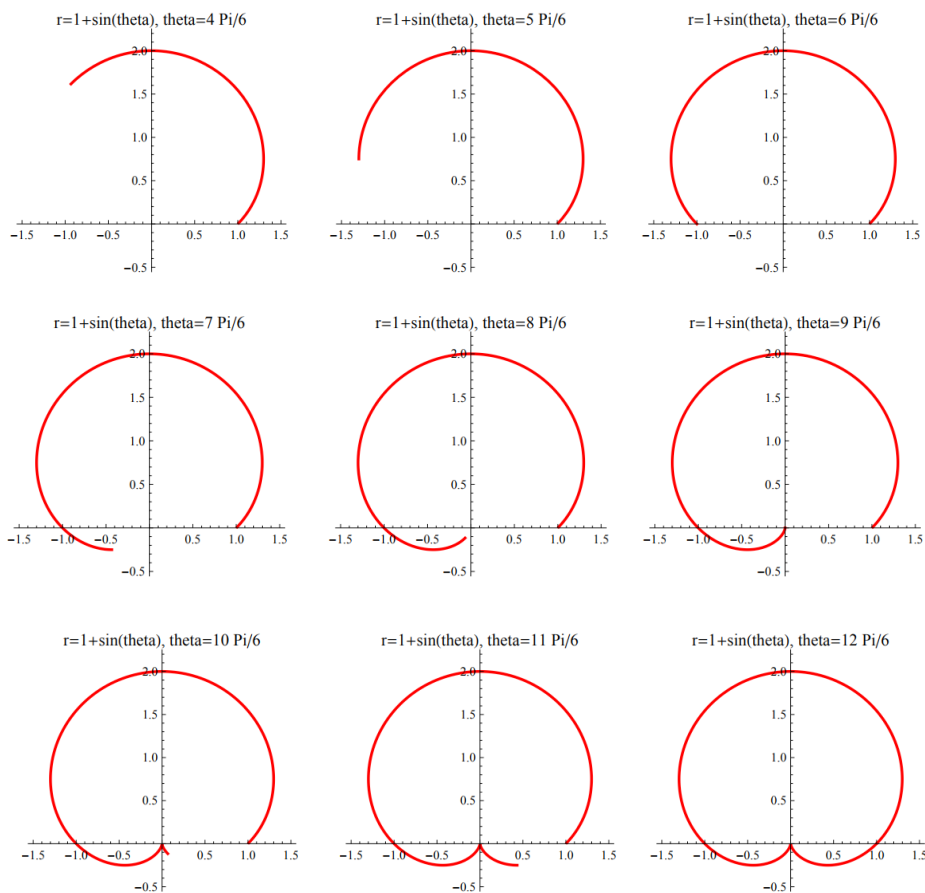
$$(x^2 + y^2 - 2x)^2 = 4(x^2 + y^2)$$



Example: Sketch the curve $r = 1 + \sin \theta$, $0 \leq \theta \leq 2\pi$ (cardioid).

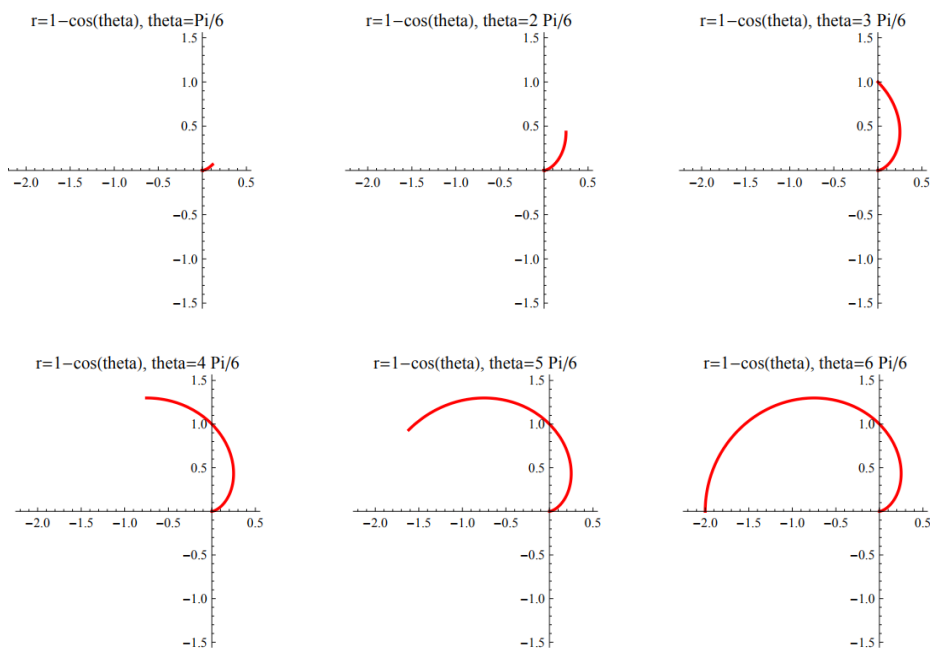
Solution: We have

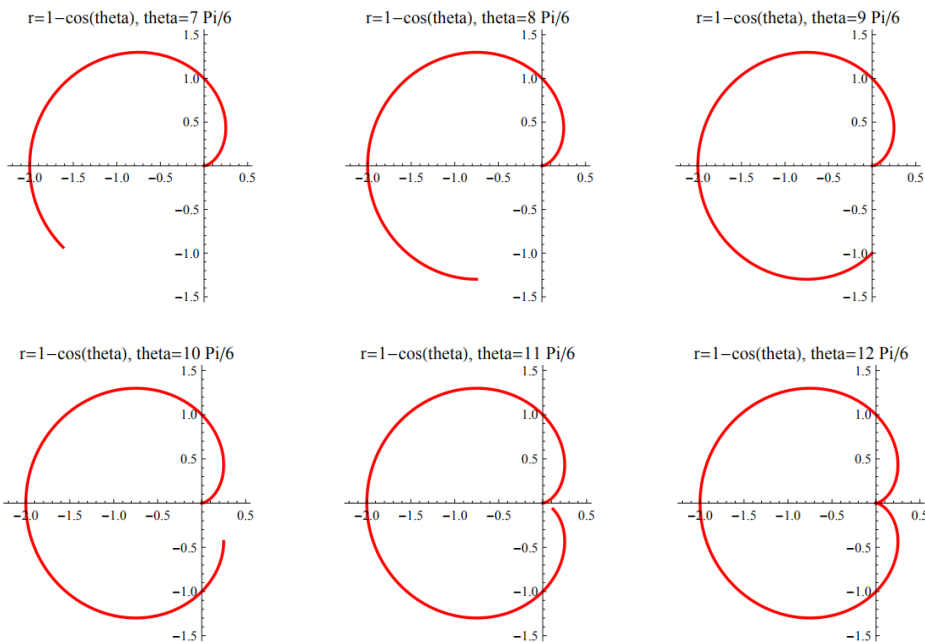




Example: Sketch the curve $r = 1 - \cos \theta, 0 \leq \theta \leq 2\pi$ (cardioid).

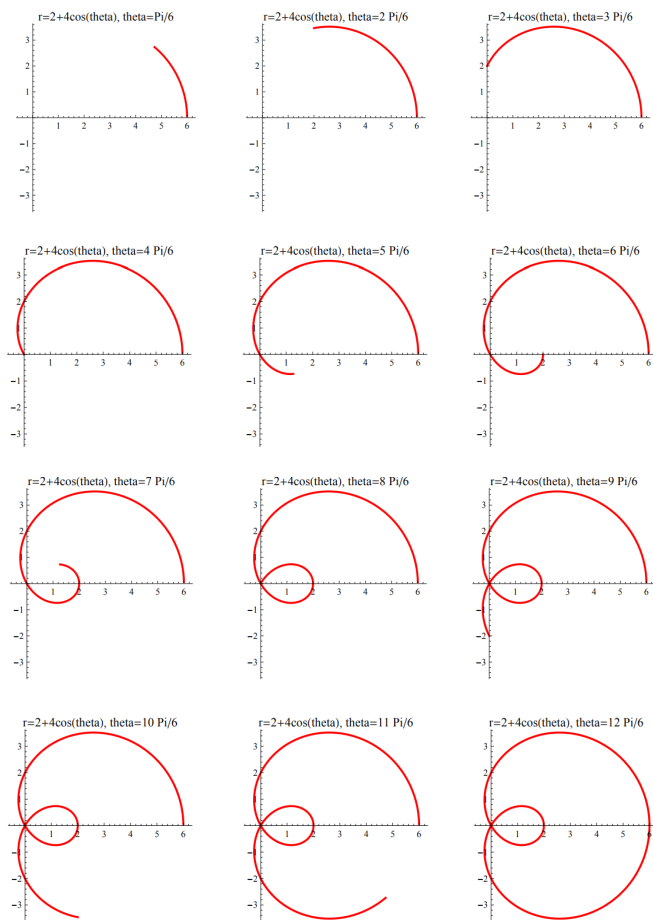
Solution: We have





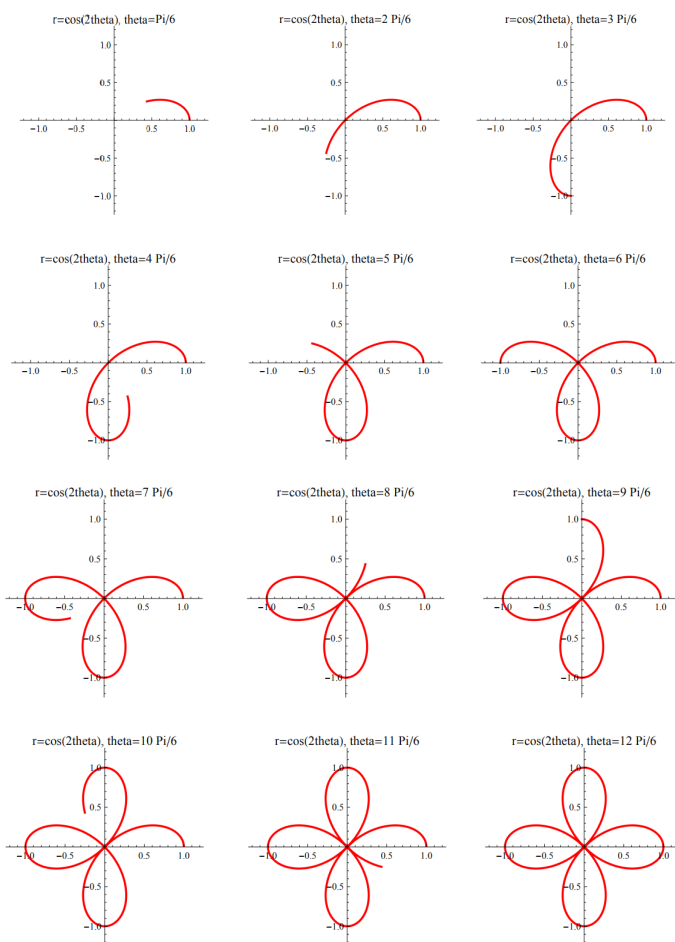
Example: Sketch the curve $r = 2 + 4 \cos \theta, 0 \leq \theta \leq 2\pi$.

Solution: We have



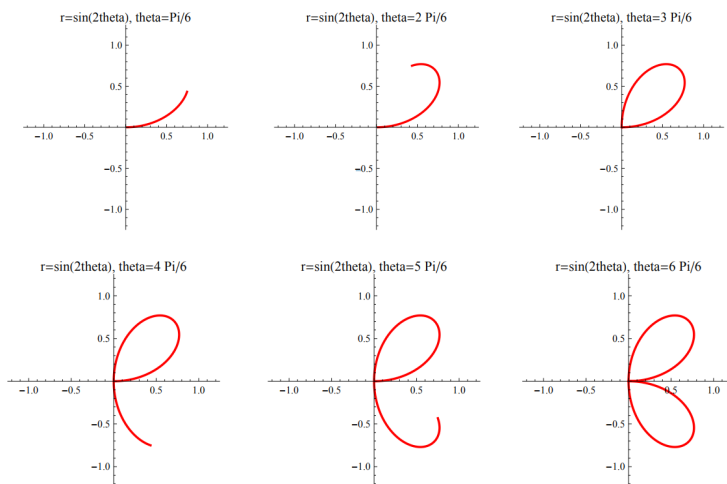
Example: Sketch the curve $r = \cos(2\theta)$, $0 \leq \theta \leq 2\pi$ (four-leaved rose).

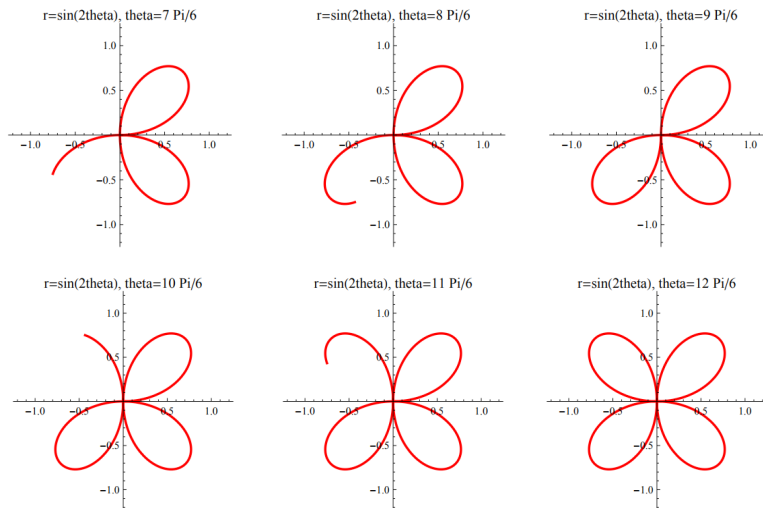
Solution: We have



Example: Sketch the curve $r = \sin(2\theta)$, $0 \leq \theta \leq 2\pi$ (four-leaved rose).

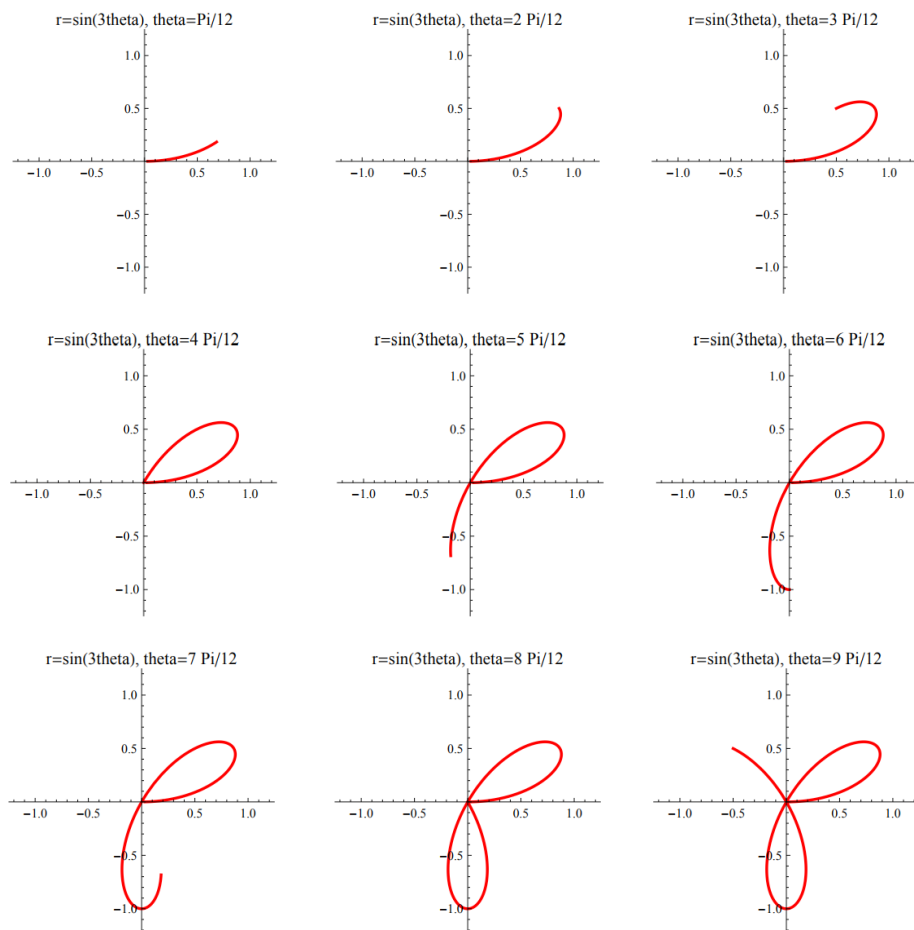
Solution: We have

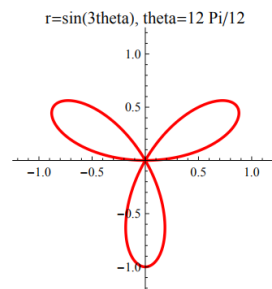
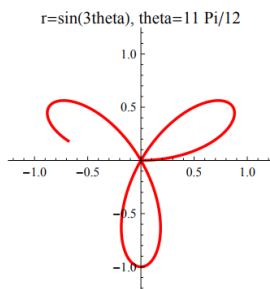
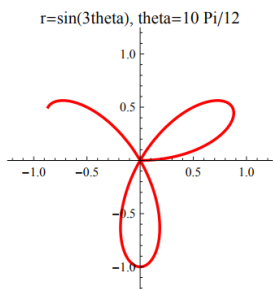




Example: Sketch the curve $r = \sin(3\theta)$, $0 \leq \theta \leq \pi$ (three-leaved rose).

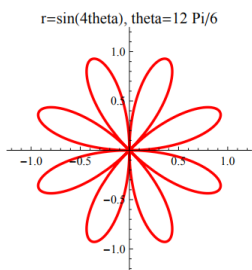
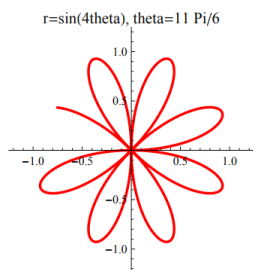
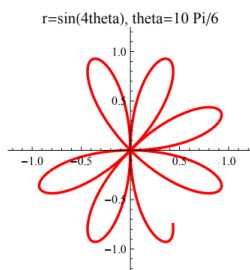
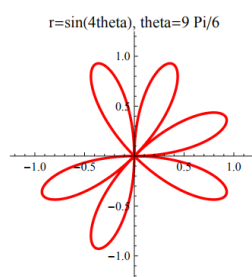
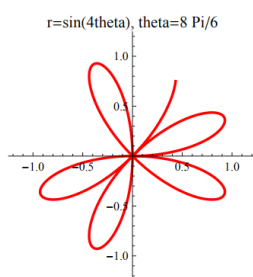
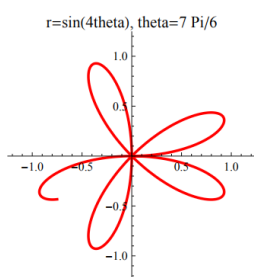
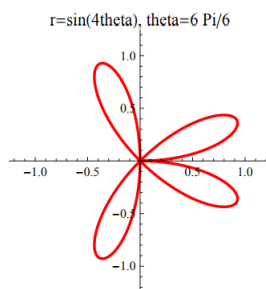
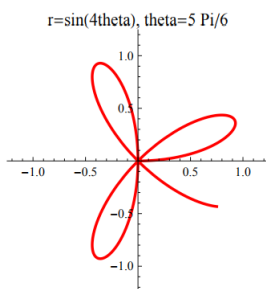
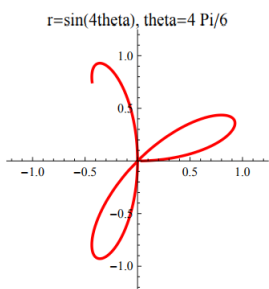
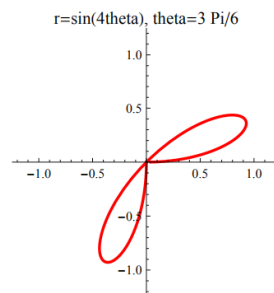
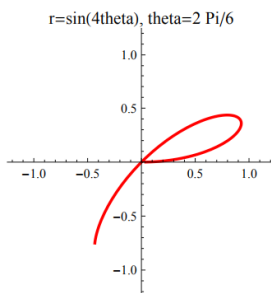
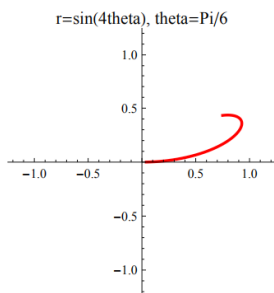
Solution: We have





Example: Sketch the curve $r = \sin(4\theta)$, $0 \leq \theta \leq 2\pi$ (eight-leaved rose).

Solution: We have



Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$ we regard θ as a parameter and write its parametric equations as

$$x = r \cos \theta = f(\theta) \cos \theta \qquad y = r \sin \theta = f(\theta) \sin \theta$$

Then, using the method for finding slopes of parametric curves and the Product Rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

Example:

(a) For the cardioid $r = 1 + \sin \theta$, find the slope of the **tangent line** when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

Solution: Using Equation 1 with $r = 1 + \sin \theta$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\cos \theta \sin \theta + (1 + \sin \theta) \cos \theta}{\cos \theta \cos \theta - (1 + \sin \theta) \sin \theta} \\ &= \frac{\cos \theta(1 + 2 \sin \theta)}{1 - 2 \sin^2 \theta - \sin \theta} = \frac{\cos \theta(1 + 2 \sin \theta)}{(1 + \sin \theta)(1 - 2 \sin \theta)} \end{aligned}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{\theta=\pi/3} &= \frac{\cos(\pi/3)(1 + 2 \sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2 \sin(\pi/3))} = \frac{\frac{1}{2}(1 + \sqrt{3})}{(1 + \sqrt{3}/2)(1 - \sqrt{3})} \\ &= \frac{1 + \sqrt{3}}{(2 + \sqrt{3})(1 - \sqrt{3})} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1 \end{aligned}$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos \theta(1 + 2 \sin \theta) = 0 \quad \text{when} \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin \theta)(1 - 2 \sin \theta) = 0 \quad \text{when} \quad \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

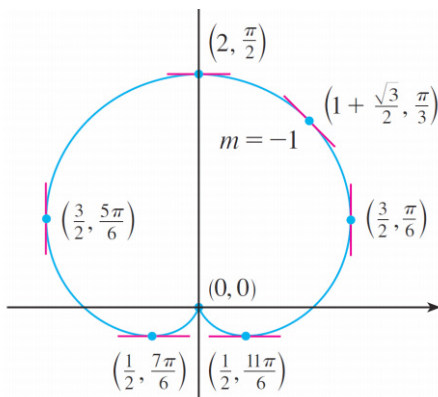
Therefore there are horizontal tangents at the points $(2, \pi/2), (\frac{1}{2}, 7\pi/6), (\frac{1}{2}, 11\pi/6)$ and vertical tangents

at $(\frac{3}{2}, \pi/6)$ and $(\frac{3}{2}, 5\pi/6)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, l' Hospital's

Tangent line to a plane curve at a given point is the straight line that “just touches” the curve at that point. Leibniz defined it as the line through a pair of infinitely close points on the curve.

Rule, we have

$$\begin{aligned}\lim_{\theta \rightarrow (3\pi/2)^-} \frac{dy}{dx} &= \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{1 + 2 \sin \theta}{1 - 2 \sin \theta} \right) \left(\lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} \right) \\ &= -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{\cos \theta}{1 + \sin \theta} = -\frac{1}{3} \lim_{\theta \rightarrow (3\pi/2)^-} \frac{-\sin \theta}{\cos \theta} = \infty\end{aligned}$$



By symmetry,

$$\lim_{\theta \rightarrow (3\pi/2)^+} \frac{dy}{dx} = -\infty$$

Thus there is a vertical tangent line at the pole.

Remark: Instead of having to remember Equation 1, we could employ the method used to derive it. For instance, in the above Example we could have written

$$x = r \cos \theta = (1 + \sin \theta) \cos \theta = \cos \theta + \frac{1}{2} \sin 2\theta$$

$$y = r \sin \theta = (1 + \sin \theta) \sin \theta = \sin \theta + \sin^2 \theta$$

Then we would have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos \theta + 2 \sin \theta \cos \theta}{-\sin \theta + \cos 2\theta} = \frac{\cos \theta + \sin 2\theta}{-\sin \theta + \cos 2\theta}$$

CONCEPT: HOW TO PLOT POINTS USING POLAR COORDINATES?

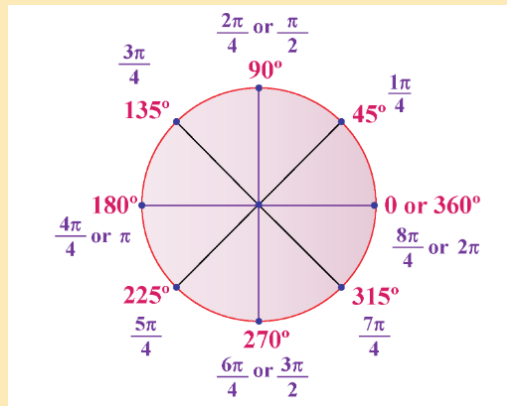
The general form for writing polar coordinates is $P(r, \theta)$

The radial coordinate is often denoted by r , and the angular coordinate by θ .

Angles in polar coordinates are expressed in either degrees or radians.

$$360^\circ = 2\pi \text{ radians}$$

A diagram for interconversion of radians and degrees is presented below for quick reference.



SUMMARY

- Polar coordinates are a two-dimensional coordinate system used to describe the position of a point in a plane using distance and angle. This system is especially useful when dealing with circular or rotational symmetrical phenomena, such as curves and patterns that revolve around a central point.
- Polar coordinates provide a method of rendering graphs and indicating the positions of points on a two-dimensional (2D) surface. The polar coordinate system is employed in mathematics, physics, engineering, navigation, robotics, and other sciences.
- In polar coordinate system, the plane is marked a point called Pole (sometimes also called Origin or Center). A horizontal ray is drawn from this point to the right; we call this the reference ray.
- Polar curve is a shape constructed using the polar coordinate system. Polar curves are defined by points that are a variable distance from the origin (the pole) depending on the angle measured off the positive x -axis.
- The last two coordinate pairs use the fact that if we end up in the opposite quadrant from the point we can use a negative r to get back to the point and of course there is both a counter clock-wise and a clock-wise rotation to get to the angle.

MULTIPLE CHOICE QUESTIONS

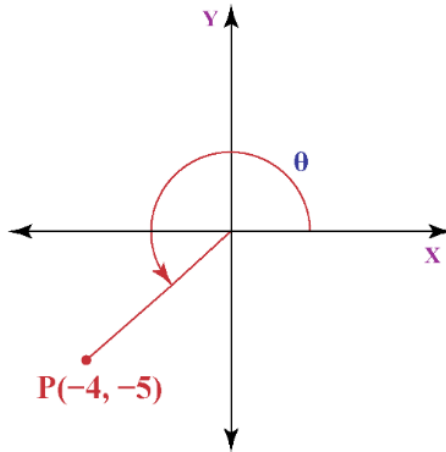
1. Polar equations of the circle for the given coordinate (x, y) which satisfies the equation given by $(x-a)^2 + (y-b)^2 = r^2$ where (a, b) is the coordinates of the centre of the circle & r is the radius.
 - a. $x = r \cos \theta, y = r \sin \theta$
 - b. $x = a + r \cos \theta, y = b + r \sin \theta$
 - c. $y = a + r \cos \theta, x = b + r \sin \theta$
 - d. $x = r \sin \theta, y = r \cos \theta$
2. The angle between Radius vector $r = a(1 - \cos \theta)$ and tangent to the curve is ϕ given by
 - a. $\phi = \left(\frac{\pi}{2}\right)$
 - b. $\phi = \pi$
 - c. $\phi = \left(-\frac{\pi}{2}\right)$
 - d. $\phi = 0$
3. Angle of intersection between two polar curves given by $r = a(1 + \sin \theta)$ & $r = a(1 - \sin \theta)$ is given by.....
 - a. $\left(\frac{\pi}{4}\right)$
 - b. $\left(\frac{\pi}{2}\right)$
 - c. Π
 - d. 0
4. What is the polar coordinate representation of the point $(3, 45^\circ)$?
 - a. $(3, \pi/4)$
 - b. $(3\sqrt{2}, \pi/4)$
 - c. $(3, \pi/2)$
 - d. $(3, \pi)$
5. Which type of curve is described by the polar equation $r = 2 + 3 \sin (\theta)$?
 - a. Circle
 - b. Parabola
 - c. Cardioid
 - d. Spiral

Answer

1. (b) 2. (a) 3. (b) 4. (b) 5. (c)

REVIEW QUESTIONS

1. Find the equation of the line $y = \frac{2}{r} = \frac{2}{\sin \theta - 3 \cos \theta}$ coordinates. We merely substitute: $r \sin \theta = 3r \cos \theta + 2$, or $r = \frac{2}{\sin \theta - 3 \cos \theta}$.
2. Test $r^2 = 9 \cos 2\theta$ for symmetry then graph its polar curve.
3. What will be Cartesian coordinates for polar coordinates $(10, 30^\circ)$?
4. What is $(15, 8)$ in polar coordinates?
5. What will be the polar coordinates for the point $(-4, -5)$?

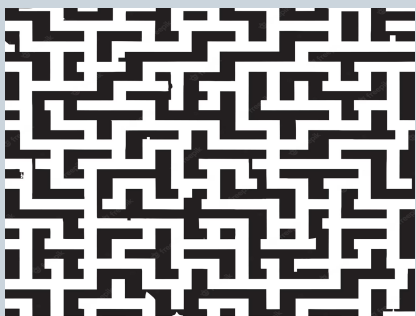


6. The Cartesian coordinates of a certain point N are $(2, 6)$. How far is the point from the pole at an angle of about 71.565° (to three decimal places)?

REFERENCES

1. Area in Polar Coordinates. (2015). [image] Available at: <http://www.emathhelp.net/notes/calculus-2/applications-of-integrals/area-in-polar-coordinates/> [Accessed 14 Jan. 2015].
2. Harcet, J., Heinrichs, L., Seiler, P. and Skoumal, M. (2012). Mathematics higher level. Oxford: Oxford University Press, p.633.
3. Polar Coordinates. (2015). [image] Available at: <http://tutorial.math.lamar.edu/Classes/CalcII/PolarCoordinates.aspx> [Accessed 14 Jan. 2015].

GEOMETRIC TRANSFORMATIONS AND CONGRUENCE



Most transformations are performed on the coordinate plane, which makes things easier to count and draw. The best way to perform a transformation on an object is to perform the required operations on the vertices of the pre-image and then connect the dots to obtain the figure. A translation is performed by moving the pre-image the requested number of spaces.

After studying this chapter, you will understand the core concepts of:

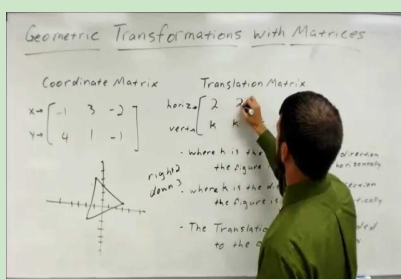
- Geometric transformations
- Congruence

INTRODUCTION

Geometric transformations and congruence are fundamental concepts in the field of geometry, playing a crucial role in understanding the relationships between shapes and figures. Geometric transformations refer to the operations that modify the position, size, and orientation of objects while preserving their underlying structure. These transformations are vital tools for exploring the symmetries and similarities present in the geometric world.

Congruence, on the other hand, is the concept of maintaining the same shape and size of an object even after undergoing a transformation. When two objects are congruent, they are essentially identical in shape and size, differing only in their position and orientation. Congruence relationships are essential in various geometrical contexts, such as in proving theorems, solving problems involving similar figures, and analyzing the properties of shapes.

There are several types of geometric transformations, each with its unique characteristics. Translation involves shifting an object's position in a straight line without altering its shape or size. Reflection reflects an object across a line, creating a mirror image. Rotation rotates an object about a fixed point, while preserving its shape and size. Scaling changes the size of an object while maintaining its proportions. These transformations can be combined to create more complex changes in figures.



CORE CONCEPT: GEOMETRIC TRANSFORMATIONS

A transformation is a general term for four specific ways to manipulate the shape of a point, a line, or shape. The original shape of the object is called the pre-image and the final shape and position of the object is the image under the transformation.

A geometric transformation is any bijection of a set having some geometric structure to itself or another such set. Specifically, "A geometric transformation is a function whose domain and range are sets of points. Most often the domain and range of a geometric transformation are both $\mathbb{R}^2$ or both $\mathbb{R}^3$. Often geometric transformations are required to be 1-1 functions, so that they have inverses."

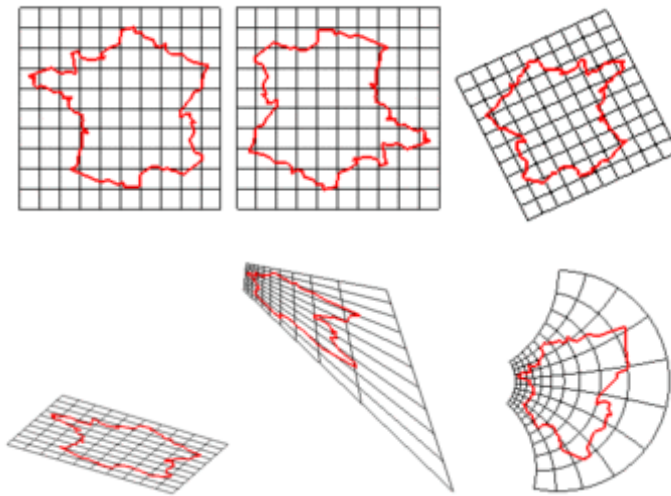
Geometric transformations can be classified by the dimension of their operand sets (thus distinguishing between planar transformations and those of space, for example).

They can also be classified according to the properties they preserve:

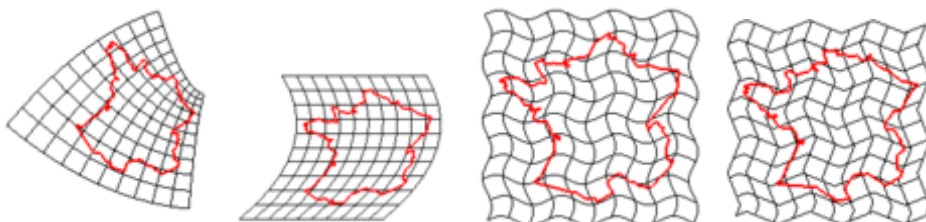
- Displacements preserve distances and oriented angles;
- Isometries preserve angles and distances
- Similarities preserve angles and ratios between distances;
- Affine transformations preserve parallelism
- Projective transformations preserve collinearity

Each of these classes contains the previous one.

- Circle inversion preserve the set of all lines and circles in the planar case (but may interchange lines and circles), and Möbius transformations conserve all planes and spheres in dimension 3.



- Diffeomorphisms (bidifferentiable transformations) are the transformations that are affine in the first order; they contain the preceding ones as special cases, and can be further refined.
- Conformal transformations, preserving angles, are, in the first order, similarities.
- equiareal transformations, preserve areas in the planar case or volumes in the three dimensional case and are, in the first order, affine transformations of determinant 1.
- Homeomorphisms (bicontinuous transformations), preserve the neighborhoods of points.



Transformations of the same type form groups that may be sub-groups of other transformation groups

Geometric transformations serve several useful roles in the study of geometries.

1. Groups of transformations can be used to classify geometries.
2. Properties of configurations that remain unchanged after applying a transformation (invariants of the transformation) are generally the significant properties that one wants to study.
3. Transformations provide the formal groundwork underlying Euclid's loose concept of superimposition.
4. Transformations allow motion to enter into the discussion of an otherwise static subject.
5. Transformations provide the link between Geometry and Abstract Algebra.

A displacement is a vector whose length is the shortest distance from the initial to the final position of a point P. It quantifies both the distance and direction of an imaginary motion along a straight line from the initial position to the final position of the point.

Transformations Methods

A transformation is an operation that changes the position of a point or shape. In some transformations the size of the shape is also altered.

Our discussion will be limited to only the following transformations: Translation, reflection, and rotation.

Translation

A translation is described by the movement of a point in the x and y directions as observed on the $x - y$ plane.

Reflection

When you look into a mirror you quickly notice a few things:

- your image body is the same distance away from the mirror as you are,
- the parts of your body are directly opposite to each other (i.e. the image nose points directly to your nose and the line joining any two corresponding points is perpendicular to the reflecting surface of the mirror)

Method of reflection:

- Plot the object point(s) on the $x - y$ plane
- Construct the **reflection** (mirror) line on the same $(x-y)$ plane and label this line (write its equation along the line).

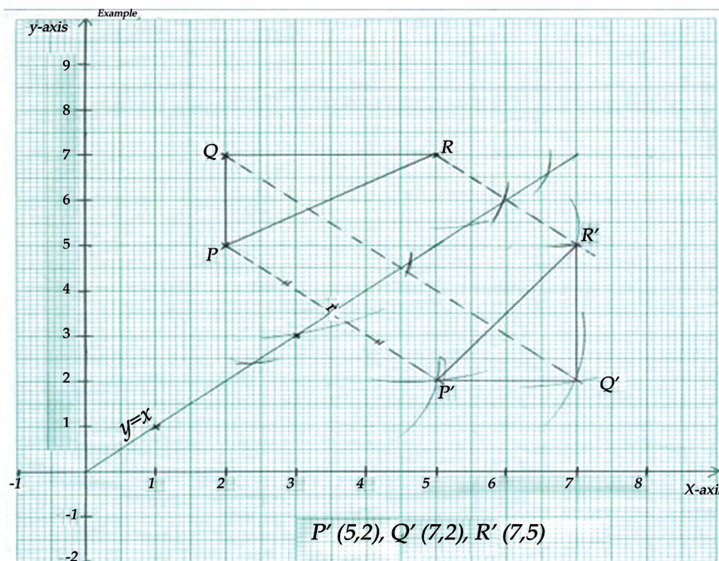
- Pick your compass from the geometry set, position the compass at the object point and stretch the radius of the compass and cut the mirror line on two different points with visible crosses.
- Without changing the radius, lift the compass and position it onto one of the crosses and make an arc on the other side of the mirror line opposite to the object. Then transfer the compass on the second cross made on the mirror line and still with the same radius, make a second arc and let it intersect with the first arc. The intersection of the arcs will be the image, I, of the object, O.

Reflection is the change in direction of a wave front at an interface between two different media so that the wave front returns into the medium from which it originated.

Example. Reflect the triangle $P(2,5), Q(2,7), R(5,7)$ through the line $y = x$.

The line $y = x$ can be drawn using the points in the table below.

x	0	1	3
y	0	1	3



Exercise :(2) Try to follow the steps in the method above)

Find the image of $R(-2, 4)$ $S(1,-3)$ $T(3, 2)$ after a reflection in the line $y = 2x + 1$

Note: When drawing the line, construct a table of values with at most three points.

x	0	1	-2
y	1	3	-3

Example. Finding the mirror line.

Given that when the triangle $A(2, 5)$ $B(7, 2)$ $C(5, 7)$ is reflected through a given line its image is $A'(5, 2)$ $B'(2, 7)$ $C'(7, 5)$. Find the mirror line and its equation

Method: - Plot both the object and image on the same $x - y$ plane.

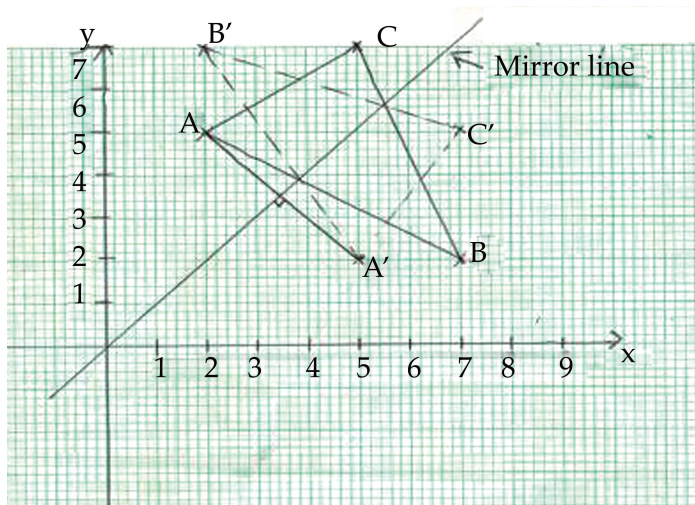
- Join one of the objects to its image (A to A')
- Bisect the line joining the object to the image

This bisector will be the mirror line.

- List down the coordinates of the midpoint of the line joining the object to the image, (AA') or any convenient point on the line (Let this be the point (x_1, y_1)).
 - Find the coordinates of the point at which the bisector cuts the y - axis, this is of the form $(0, c)$. Let this be the point (x_2, y_2) .
- Calculate the gradient of this line, m , from:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Use the general equation of a line $y = mx + c$



From above using (x_1, y_1) as $(3, 3)$ and (x_2, y_2) as $(0, 0)$

$$m = \frac{0 - 3}{0 - 3} = \frac{-3}{-3} = 1$$

Using $y = mx + c$

$$\therefore y = 1(x) + 0$$

$y = x$ (The equation of the mirror line)

Summary of common reflections

Mirror line	Nature of image as compared to object
a) $x - \text{axis}$	- The x remains the same but y changes sign $(2, -3) \rightarrow (2, 3)$
d) $y - \text{axis}$	- The y remains the same but x changes sign $(4, 5) \rightarrow (-4, 5)$
c) $x = y$	- The x becomes the new y and the y becomes the new x $(7, 3) \rightarrow (3, 7)$
d) $x = -y$ or $x + y = 0$	- The negative of x becomes the new y and the negative of y becomes the new x . $(6, 9) \rightarrow (-9, -6)$ Note. This summary should be used to help you check your answers and not as a substitute to graphical work.

Exercise: (3) R is a reflection in the line $x + y = 0$. T is a translation which maps $(-1, -1)$ onto $(2, 0)$. Find the image of $A(0, 2), B(2, 2), C(0, 4)$ under (i) RT^2 (ii) T^3 .

Rotation

A rotation is a turn described by the angle turned through (angle of rotation) and the point at which the turning is done (Centre of rotation). The angle of rotation may be positive (turn in the anticlockwise sense) or negative (turn in the clockwise sense)

Method of rotation:

- Plot the object points and the center of rotation on the same $x - y$ plane.
- Join A to the center of rotation and extend the line slightly. Measure the distance of A from the center of rotation
- Place your protractor at the center of rotation along this line running from A to the center so as to measure the required angle (take note of the positioning of the protractor to be able to measure positive and negative angles)
- Measure the angle and remove the protractor. Trace out the line showing the measured angle. (This is the image line).
- Measure off the same distance as that of the object from the center, along the image line. (Using a compass, place it at the center and stretch it's radius up to the object. Turn the compass without lifting it off the center and cut off the same radius along the image line). This will be the image point.

- Repeat (b) to (e) for all the other points.

How to Perform Transformations

Translation

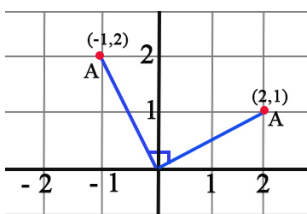
The process of translation is similar to shifting or sliding an object. In two dimensions, the representation for the factor of translation is $T(+a, +b)$, where a denotes the shift in the direction of X axis, while b shows shift in the direction of Y axis. For instance, if a point $(3, -4)$ has to be translated by $(+1, -2)$, it would be shifted to the point $(3 + 1, -4 - 2)$ or $(4, -6)$. In this way, the point $(3, -4)$ is translated into point $(4, -6)$.

Rotation

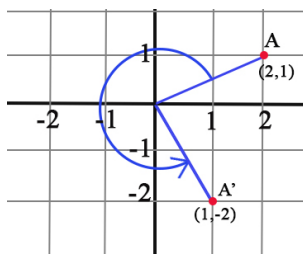
In order to perform common rotations such as rotating a point by 90° , 180° , 270° etc., we may use an easy trick. All we have to do is, just locate the given point (called pre-image) on graph paper and then physically rotate the graph paper, the new location that we obtain for the given point is denotes the coordinates of rotated point (called image).

Usually, the rotation is denoted by $R(\text{center}, \text{angle})$, where “center” represents the center of rotation, i.e. the point about which the rotation is performed and “angle” stands for the angle in degrees to which the pre-image is to be rotated.

The images down below show rotation of point $(2, 1)$ by 90° and 270° respectively.



and



Reflection

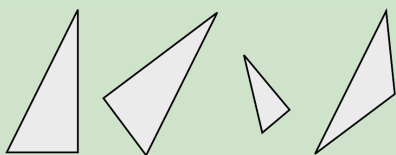
In general, the reflection of a point is the location of this point if a given line is considered as the mirror. The reflection in the X-axis can be given by the following general rule –

A reflection of point (x, y) in X-axis is $(x, -y)$. For example : a reflection of $(3, 4)$ in X-axis would be the point $(3, -4)$. On the other hand, the rule for reflecting a point (x, y) in Y-axis is $(-x, y)$. For instance: a reflection of $(3, 4)$ in Y-axis would be the point $(-3, 4)$. Also, a reflection of a point (x, y) in the line $y = x$ can be observed by the point (y, x) . We can say that a reflection of $(3, 4)$ in line $y = x$ would be the point $(4, 3)$.

Dilation

The dilation of a point is performed by dividing by multiplying its coordinates by a given factor. An image can be shirked by dividing the coordinates of all its points by a factor, while it can be stretched by multiplying the **coordinates** of all its points by a given factor.

Coordinates are often chosen such that one of the numbers represents a vertical position, and two or three of the numbers represent a horizontal position. A common choice of it is latitude, longitude and elevation.



CORE CONCEPT: CONGRUENCE

Two objects are congruent if they have the same dimensions and shape. Very loosely, you can think of it as meaning 'equal', but it has a very precise meaning that you should understand completely, especially for complex shapes such as polygons.

In geometry, two figures or objects are congruent if they have the same shape and size, or if one has the same shape and size as the mirror image of the other.

More formally, two sets of points are called congruent if, and only if, one can be transformed into the other by an isometry, i.e., a combination of rigid motions, namely a translation, a rotation, and a reflection. This means that either object can be repositioned and reflected (but not resized) so as to coincide precisely with the other object. So two distinct plane figures on a piece of paper are congruent if we can cut them out and then match them up completely. Turning the paper over is permitted.

In elementary geometry the word congruent is often used as follows. The word equal is often used in place of congruent for these objects.

- Two line segments are congruent if they have the same length.
- Two angles are congruent if they have the same measure.
- Two circles are congruent if they have the same diameter.

In this sense, two plane figures are congruent implies that their corresponding

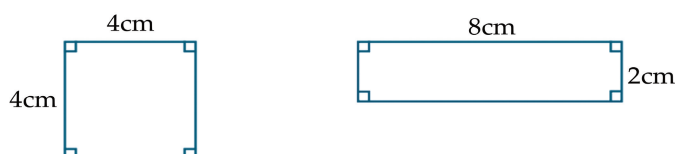
Diagonal is a line segment joining two vertices of a polygon or polyhedron, when those vertices are not on the same edge. Informally, any sloping line is called diagonal.

characteristics are «congruent» or “equal” including not just their corresponding sides and angles, but also their corresponding **diagonals**, perimeters and areas.

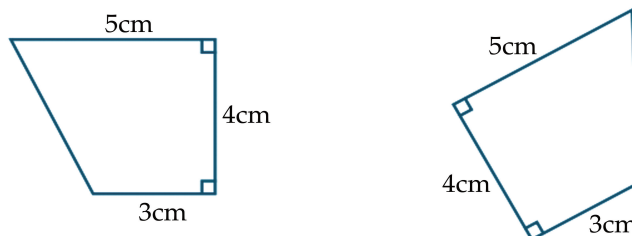
The related concept of similarity applies if the objects have the same shape but do not necessarily have the same size. (Most definitions consider congruence to be a form of similarity, although a minority require that the objects have different sizes in order to qualify as similar.)

Motivation

Two geometric figures may resemble each other in some ways, but differ in others. For example, all the angles of the square and the rectangle below are right angles, and they have the same area, but their side lengths are different.



On the other hand, the two figures below are exactly the same in all respects apart from their position and orientation – we can pick up one of them and place it so that it fits exactly on top of the other. Such figures are called congruent.



Knowing that two figures are congruent is important. For example, if we measure or calculate the unmarked side length of the diagram on the left above, then the matching length is the same in the diagram on the right above. (Pythagoras’ theorem gives us the answer 25 cm for this length.) This very simple idea of matching lengths, matching angles, and matching areas becomes the means by which we can prove many geometric results. A polygon can always be divided up into triangles, so that arguments about the congruence of polygons can almost always be reduced to arguments about congruent triangles. Most of our discussion therefore concerns congruent triangles. We shall develop the four standard tests used to check that

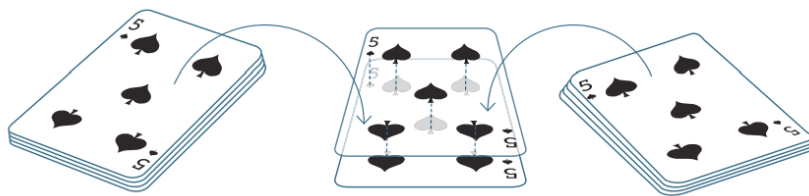
two triangles are congruent. It is also true that figures involving curves can be congruent, such as circles of the same radius. A good way to think about congruence is to ask, ‘How much information do need to give someone about a figure if they are going to draw it?’ For example, surveyors go to a lot of trouble taking careful measurements of a landscape. They must know that everything important about that landscape can be calculated from the measurements that they have taken. In an analogous way, a certain minimum amount of information is needed to draw a particular triangle. We will develop the congruence

Two angles are congruent if they have the same measure. So if two separate angles have measures of 30° and 23° for example, they are not congruent because they have different measures. Congruent angles may lie in different orientations or positions.

tests as a solution to this question. A great deal of mathematics depends on finding and exploiting symmetries. This is particularly true in geometry, where the elementary figures that we study —like squares, rectangles, circles – exhibit obvious reflection and rotation symmetries. Argument based on direct appeals to symmetry, however, is notoriously difficult to construct and to evaluate, and the ancient Greek mathematicians, most famously Euclid, introduced argument based on congruence as a replacement. The resulting geometric proofs, using mostly only congruent triangles, are clear and straightforward in their logic. Justifying the congruence tests is no easy matter – the logical problems inherent in them were only sorted out at the end of the nineteenth century, by David Hilbert in particular. We will take the first three tests as axioms of our geometry, and will be content to demonstrate their reasonableness using ruler-and compasses constructions.

Congruence of Plane Figures

If we take the five of spades from each of two identical decks of cards, they look exactly the same. We can move one card and place it on top of the other one so that the pictures on the two cards coincide exactly, as shown below.



Two objects like this are called congruent. The word ‘congruent’ comes from Latin and means ‘in agreement’ or ‘in harmony’.

Definition: Two plane figures are called congruent if one figure can be moved so that it fits exactly on top of the other figure. If two figures are congruent, such a movement can always be done by a sequence of translations, rotations and reflections reflect the first figure in any axis if it has the opposite parity to the second, then translate any point of the first figure to the matching point of the second figure, then rotate the first figure until it fits exactly on top of the second. Thus congruent figures

Interval is a set of real numbers with the property that any number that lies between two numbers in the set is also included in the set.

can be defined in an alternative way that specifies the allowed transformations: ‘Two plane figures are called congruent if one can be moved by a sequence of translations, rotations and reflections so that it fits exactly on top of the other figure.’ The example above with the cards involved translations and rotations. In the diagram to the right, the two footprints are congruent because one can be reflected onto the other.



Matching Up the Parts of Congruent Figures

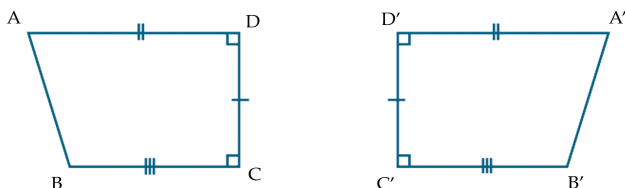
When two figures are congruent, we can match every part of one figure with the corresponding part of the other figure. For example, the two figures below are congruent. Some of the matching's of points with points, **intervals** with intervals, and angles with angles are:

$$A \leftrightarrow A'$$

$$AB \leftrightarrow A'B'$$

$$\angle BAD \leftrightarrow \angle B'A'D'$$

Matching intervals have the same length, and matching angles have the same size. Also the matching regions $ABCD$ and $A'B'C'D'$ have the same area.



Determining Congruence of Polygons

For two polygons to be congruent, they must have an equal number of sides (and hence an equal number—the same number—of vertices). Two polygons with n sides are congruent if and only if they each have numerically identical sequences (even if clockwise for one polygon and counterclockwise for the other) side-angle-side-angle-... for n sides and n angles.

Congruence of polygons can be established graphically as follows:

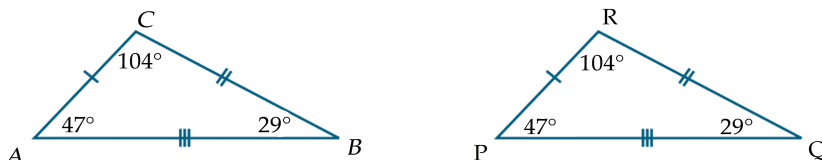
- First, match and label the corresponding vertices of the two figures.
- Second, draw a vector from one of the vertices of the one of the figures to the corresponding vertex of the other figure. *Translate* the first figure by this vector so that these two vertices match.
- Third, *rotate* the translated figure about the matched vertex until one pair of corresponding sides matches.
- Fourth, *reflect* the rotated figure about this matched side until the figures match.

Two triangles are congruent if their corresponding sides are equal in length, in which case their corresponding angles are equal in measure.

If at any time the step cannot be completed, the polygons are not congruent.

Congruent Triangles

When geometric figures are made up entirely of intervals, rather than curves, arguments about congruence can usually be reduced to arguments about congruent triangles. This is one reason why so much attention is given to congruent triangles. Here are two congruent triangles. The triangle PQR on the right has been formed by a translation of the triangle ABC on the left.



The vertices and sides of triangle ABC are matched by the congruence with the vertices and sides of triangle PQR as follows:

$$\begin{array}{lll}
 A \leftrightarrow P & B \leftrightarrow Q & C \leftrightarrow R \\
 BC \leftrightarrow QR & CA \leftrightarrow RP & AB \leftrightarrow PQ
 \end{array}$$

All this can be expressed extremely concisely by using the symbol $\equiv$ for 'is congruent to', and by observing the convention that the two triangles are named with the vertices in matching order:

$$ABC \equiv PQR$$

Determining congruence

Sufficient evidence for congruence between two triangles in Euclidean space can be shown through the following comparisons:

Hypotenuse is the side of a right triangle that's opposite the 90-degree angle.

It's a term specific to math, specifically geometry.

- **SAS** (Side-Angle-Side): If two pairs of sides of two triangles are equal in length, and the included angles are equal in measurement, then the triangles are congruent.
- **SSS** (Side-Side-Side): If three pairs of sides of two triangles are equal in length, then the triangles are congruent.
- **ASA** (Angle-Side-Angle): If two pairs of angles of two triangles are equal in measurement, and the included sides are equal in length, then the triangles are congruent. The ASA Postulate was contributed by Thales of Miletus (Greek). In most systems of axioms, the three criteria – SAS, SSS and ASA – are established as theorems. In the School Mathematics Study Group system **SAS** is taken as one (#15) of 22 postulates.
- **AAS** (Angle-Angle-Side): If two pairs of angles of two triangles are equal in measurement, and a pair of corresponding non-included sides are equal in length, then the triangles are congruent. AAS is equivalent to an ASA condition, by the fact that if any two angles are given, so is the third angle, since their sum should be 180° . ASA and AAS are sometimes combined into a single condition, **AAcorrS** – any two angles and a corresponding side.
- **RHS** (Right-angle-Hypotenuse-Side), also known as **HL** (Hypotenuse-Leg): If two right-angled triangles have their **hypotenuses** equal in length, and a pair of shorter sides are equal in length, then the triangles are congruent.

Side-side-angle

The SSA condition (Side-Side-Angle) which specifies two sides and a non-included angle (also known as ASS, or Angle-Side-Side) does not by itself prove congruence. In order to show congruence, additional information is required such as the measure of the corresponding angles and in some cases the lengths of the two pairs of corresponding sides. There are a few possible cases:

If two triangles satisfy the SSA condition and the length of the side opposite the angle is greater than or equal to the length of the adjacent side (SsA, or long side-short side-angle), then the two triangles are congruent. The opposite side is sometimes

longer when the corresponding angles are acute, but it is *always* longer when the corresponding angles are right or obtuse. Where the angle is a right angle, also known as the Hypotenuse-Leg (HL) postulate or the Right-angle-Hypotenuse-Side (RHS) condition, the third side can be calculated using the Pythagorean Theorem thus allowing the SSS postulate to be applied.

If two triangles satisfy the SSA condition and the corresponding angles are acute and the length of the side opposite the angle is equal to the length of the adjacent side multiplied by the sine of the angle, then the two triangles are congruent.

If two triangles satisfy the SSA condition and the corresponding angles are acute and the length of the side opposite the angle is greater than the length of the adjacent side multiplied by the sine of the angle (but less than the length of the adjacent side), then the two triangles cannot be shown to be congruent. This is the ambiguous case and two different triangles can be formed from the given information, but further information distinguishing them can lead to a proof of congruence.

Angle-angle-angle

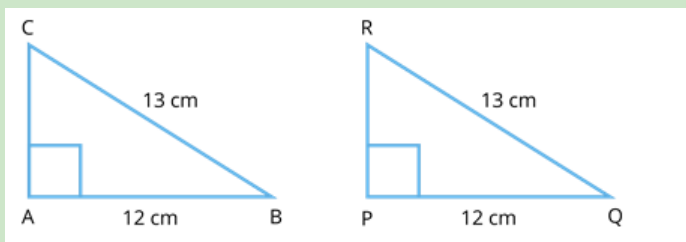
In Euclidean geometry, AAA (Angle-Angle-Angle) (or just AA, since in Euclidean geometry the angles of a triangle add up to 180°) does not provide information regarding the size of the two triangles and hence proves only similarity and not congruence in Euclidean space. However, in spherical geometry and hyperbolic geometry (where the sum of the angles of a triangle varies with size) AAA is sufficient for congruence on a given curvature of surface.

CONCEPT: RIGHT ANGLE-HYPOTENUSE-SIDE (RHS) CONGRUENCE CRITERION

The RHS congruence criterion is similar to the SAS congruence criterion.

The only difference is that 90° is the only angle used.

To prove that the two right angled triangles are congruent, it is enough to prove that any one side and the hypotenuse one triangle is congruent to the corresponding side and the hypotenuse of the other triangle.



From the figure given above, sides $AB=PQ=12$ cm, and the hypotenuses $BC=QR=13$ cm.

Since the corresponding sides and the hypotenuses are equal, the triangles $\triangle ABC$ and $\triangle PQR$ are congruent.

SUMMARY

- Geometric transformations and congruence are fundamental concepts in the field of geometry, playing a crucial role in understanding the relationships between shapes and figures. Geometric transformations refer to the operations that modify the position, size, and orientation of objects while preserving their underlying structure.
- A geometric transformation is any bijection of a set having some geometric structure to itself or another such set. Specifically, “A geometric transformation is a function whose domain and range are sets of points.
- Two objects are congruent if they have the same dimensions and shape. Very loosely, you can think of it as meaning ‘equal’, but it has a very precise meaning that you should understand completely, especially for complex shapes such as polygons.
- Two plane figures are congruent implies that their corresponding characteristics are “congruent” or “equal” including not just their corresponding sides and angles, but also their corresponding diagonals, perimeters and areas.
- The SSA condition (Side-Side-Angle) which specifies two sides and a non-included angle (also known as ASS, or Angle-Side-Side) does not by itself prove congruence.
- AAA (Angle-Angle-Angle) (or just AA, since in Euclidean geometry the angles of a triangle add up to 180°) does not provide information regarding the size of the two triangles and hence proves only similarity and not congruence in Euclidean space.

MULTIPLE CHOICE QUESTIONS

1. Which geometric transformation preserves the shape and size of an object while changing its position in a straight line?
 - a. Reflection
 - b. Rotation
 - c. Translation
 - d. Scaling
2. Congruent figures are objects that have the same:
 - a. Size
 - b. Shape
 - c. Size and shape
 - d. Orientation
3. Which geometric transformation creates a mirror image of an object across a line?
 - a. Rotation
 - b. Scaling
 - c. Translation
 - d. Reflection
4. If two triangles are congruent, which of the following statements is true?
 - a. They have the same size and shape.
 - b. They have the same size but different shapes.
 - c. They have the same shape but different sizes.
 - d. They have different sizes and shapes.
5. What does a rotation transformation preserve in a geometric figure?
 - a. Shape
 - b. Size
 - c.. Both shape and size
 - d. Neither shape nor size

Answer

1. (c) 2. (c) 3. (d) 4. (a) 5. (c)

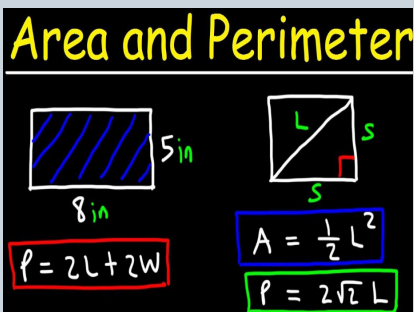
REVIEW QUESTIONS

1. Define the useful methods of transformations.
2. Illustrate the perception and motivation of congruence.
3. What are the congruence of plane figures?
4. Regulate the congruence of polygons.
5. What do you understand by the congruent triangles?

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PERIMETER AND AREA



The perimeter of a two-dimensional shape is the distance around its outer boundary. It can be thought of as the length of the path one would need to travel along the edges of the shape to encompass its entire outline. The area of a two-dimensional shape is the measure of the amount of space enclosed by the shape's boundary. It is a measure of the extent or size of the surface within the shape's edges.

After studying this chapter, you will understand the core concepts of:

- Concept of Perimeter and Area
- Surface Area and Volume

INTRODUCTION

Area and Perimeter is an important topic in Mathematics, which is used in everyday life. This is applicable to any shape and size whether it is regular or irregular. Every shape has its own area and perimeter formula.

The area is the region bounded by the shape of an object. The space covered by the figure or any two-dimensional geometric shape, in a plane, is the area of the shape. The area of all the shapes depends upon their dimensions and properties. Different shapes have different areas. The area of the square is different from the area of a kite.

Perimeter of a shape is defined as the total distance around the shape. Basically, perimeter is the length of any shape if it is expanded in a linear form. A perimeter is a total distance that encompasses a shape, in a 2d plane. The perimeter of different shapes can match in length with each other depending upon their dimensions.



CORE CONCEPT: CONCEPT OF PERIMETER AND AREA

Area and Perimeter are two important concepts. Sometimes, the term 'Area' is confused with 'Perimeter'. They both are entirely different. The area is defined as the amount of space occupied by any two-dimensional shape. Perimeter defines the boundary or outline of a flat shape.

A perimeter is a path that surrounds a two-dimensional shape. The term may be used either for the path or its length it can be thought of as the length of the outline of a shape. The perimeter of a circle or ellipse is called its circumference.

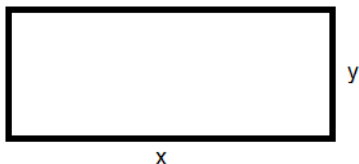
The perimeter is the length of the outline of a shape. To find the perimeter of a rectangle or square you have to add the lengths of all the four sides. x is in this case the length of the rectangle while y is the width of the rectangle.

The perimeter, P , is:

$$P = x + x + y + y$$

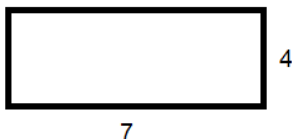
$$P=2x+2y$$

$$P=2(x+y)$$



Example

Find the perimeter of this rectangle:



$$P=7+7+4+4$$

$$P=2\cdot 7+2\cdot 4$$

$$P=2\cdot (7+4)$$

$$P=2\cdot 11$$

$$P=22\text{in}$$

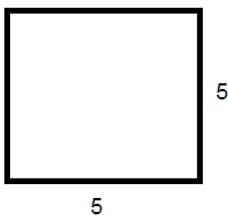
A part of the plane enclosed by a simple closed figure is called a plane region and the measurement of plane region enclosed is called its area. The area is measurement of the surface of a shape. To find the area of a rectangle or a square you need to multiply the length and the width of a rectangle or a **square**.

Area, A , is x times y .

$$A=x\cdot y$$

Example

Find the area of this square.



$$A=x\cdot y$$

Square is a regular quadrilateral, which means that it has four equal sides and four equal angles.

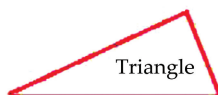
$$A = 5 \cdot 5$$

$$A = 25 \text{ in}^2$$

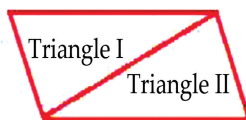
There are different units for perimeter and area. The perimeter has the same units as the length of the sides of rectangle or square whereas the area's unit is squared.

Area of a Triangle

The area of a polygon is the number of square units inside that polygon. Area is 2-dimensional like a carpet or an area rug. A triangle is a three-sided polygon.



To find the area of a triangle, multiply the base by the height, and then divide by 2. The division by 2 comes from the fact that a parallelogram can be divided into 2 triangles. For example, in the diagram to the left, the area of each triangle is equal to one-half the area of the parallelogram.



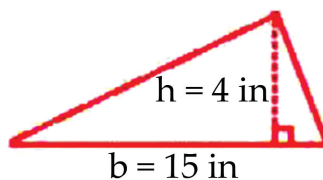
Since the area of a parallelogram is $A = b \cdot h$, the area of a triangle must be one-half the area of a parallelogram. Thus, the formula for the area of a triangle is:

$$A = \frac{1}{2} \cdot b \cdot h \text{ or } A = \frac{b \cdot h}{2}$$

where b is the base, h is the height and $\cdot$ means multiply.

The base and height of a triangle must be perpendicular to each other. In each of the examples below, the base is a side of the triangle. However, depending on the triangle, the height may or may not be a side of the triangle. For example, in the right triangle in Example 2, the height is a side of the triangle since it is perpendicular to the base. In the triangles in Examples 1 and 3, the lateral sides are not perpendicular to the base, so a dotted line is drawn to represent the height.

Example 1: Find the area of an acute triangle with a base of 15 inches and a height of 4 inches.



Solution:

$$A = \frac{1}{2} \cdot b \cdot h$$

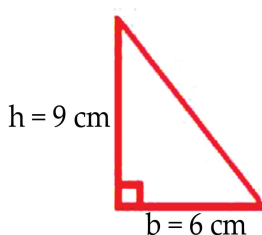
$$A = \frac{1}{2} \cdot (15 \text{ in}) \cdot (4 \text{ in})$$

$$A = \frac{1}{2} \cdot (6 \text{ in}^2)$$

$$\ddot{u} = 30 \quad ^2$$

The height must be perpendicular to the base. If necessary, draw an altitude from the vertex opposite the base to help you determine the height.

Example 2: Find the area of a right triangle with a base of 6 centimeters and a height of 9 centimeters.



Solution:

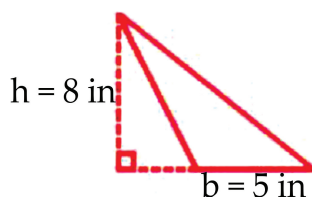
$$A = \frac{1}{2} \cdot b \cdot h$$

$$A = \frac{1}{2} \cdot (6 \text{ cm}) \cdot (9 \text{ cm})$$

$$A = \frac{1}{2} \cdot (54 \text{ cm}^2)$$

$$A = 27 \text{ cm}^2$$

Example 3: Find the area of an obtuse triangle with a base of 5 inches and a height of 8 inches.



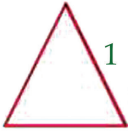
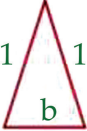
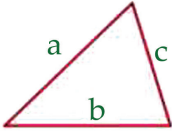
Solution:

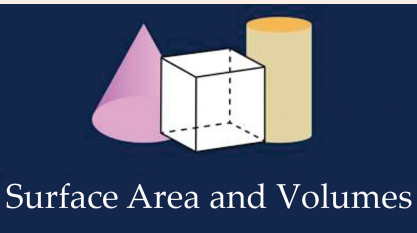
$$A = \frac{1}{2} \cdot b \cdot h$$

$$A = \frac{1}{2} \cdot (5\text{in}) \cdot (8\text{in})$$
$$A = \frac{1}{2} \cdot (40\text{in}^2)$$
$$A = 20\text{in}^2$$

Perimeter of a Triangle

The perimeter of a triangle equals the sum of the length of its three sides. The perimeter is the total distance around the outside

Equilateral Triangle	Isosceles Triangle	Scalene Triangle
$P = 3 \cdot l$	$P = 2 \cdot l + b$	$P = a + b + c$
		



CORE CONCEPT: SURFACE AREA AND VOLUME

Surface area of a solid object is a measure of the total area that the surface of the object occupies. Volume is the quantity of three-dimensional space enclosed by a closed surface, for example, the space that a substance or shape occupies or contains.

Cylinder is the surface formed by the points at a fixed distance from a given straight line called the axis of the cylinder.

Volume and surface area are two related concepts in the study of mathematics. They’re both important to understand, but equally important is understanding how they differ and what they mean. This is especially the case when it comes to computing the volume and surface areas of a prism or a **cylinder**.

If you think of wrapping a present in a box, you can get a good sense of how volume and surface area differ. First, you have to consider the size of the box, when you consider the size of the present. How much interior space does your box need to have so a present will fit? The measurement of the box’s capacity, how much it will hold, is its volume. Next you have to wrap the present. The amount of wrapping paper, which will cover the exterior of the box, is a very different calculation than

the capacity of the box. You'll need a separate measurement or some good guessing, to figure out the sum of the sides of all the surfaces or the surface area.

Volume of a square or rectangular box is pretty easy to compute. Simply multiply height times length times width to get the measurement. With a square it's even easier, you merely cube one side's length, since they all measure the same. If the side length is a , the formula is $a \times a \times a$ or a^3 . When you are comparing volume and surface area, you'll note a very different formula. You need to get the area of each face, and then add the areas of all faces together. With a square prism or cube, you'd essentially compute the area $a \times a$ or a^2 , multiplied by 6 ($6a^2$). When you're working with a rectangular prism, you'll have to the area the of 3 pairs of equal sides, which needed to be added together to determine surface area.

Work on volume and surface area are differ a little when you are trying to calculate the area of a cylinder. The formula for a volume of a cylinder is the area of one circular face multiplied times the height of the cylinder. It reads: $\pi r^2 \times h$, or pi times the radius squared times height. Getting the surface area of the cylinder is a little trickier since the circular portion is essentially one continuous face. Computing surface area of a cylinder means computing the lateral area of this face.

Lateral area formula is the following $\pi r^2 r$ or πd (pi times the radius doubled or pi times the diameter), multiplied to the height, $\pi r^2 \times h$. This is essentially the circumference of one circle times the height of the cylinder. To compute the entire formula you also need to add in the top and bottom circular faces' areas. Since in a cylinder these are equal, the formula is $2\pi r^2$. This calculation is then added to the lateral area to compute the whole surface area in the following expression:

$$\pi r^2 r \times h + 2\pi r^2 = \text{lateral area.}$$

Polyhedron

A polyhedron is a solid object whose surface is made up of a number of flat faces which themselves are bordered by straight lines. Each face is in fact a polygon, a closed shape in the flat 2-dimensional plane made up of points joined by straight lines.

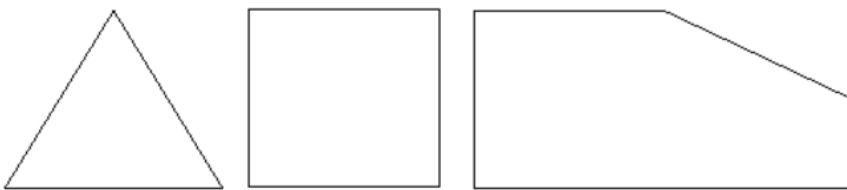


Figure 1: The familiar triangle and square are both polygons, but polygons can also have more irregular shapes like the one shown on the right.

Polygons are not allowed to have holes in them, as the figure below illustrates: the left-hand shape here is a polygon, while the right-hand shape is not.

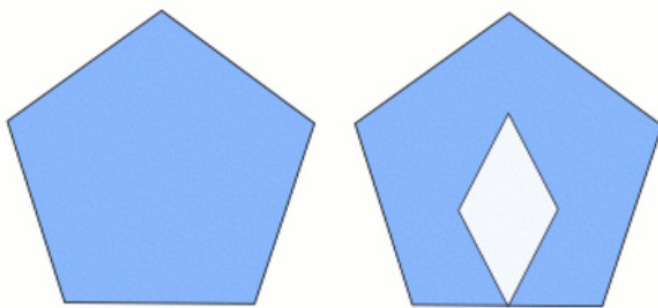


Figure 2: The shape on the left is a polygon, but the one on the right is not, because it has a ‘hole’.

A polygon is called regular if all of its sides are the same length, and all the angles between them are the same; the triangle and square in figure 1 and the pentagon in figure 2 are regular.

A polyhedron is what you get when you move one dimension up. It is a closed, solid object whose surface is made up of a number of polygonal faces. We call the sides of these faces edges two faces meet along each one of these edges. We call the corners of the faces vertices, so that any vertex lies on at least three different faces. To illustrate this, here are two examples of well-known polyhedra.

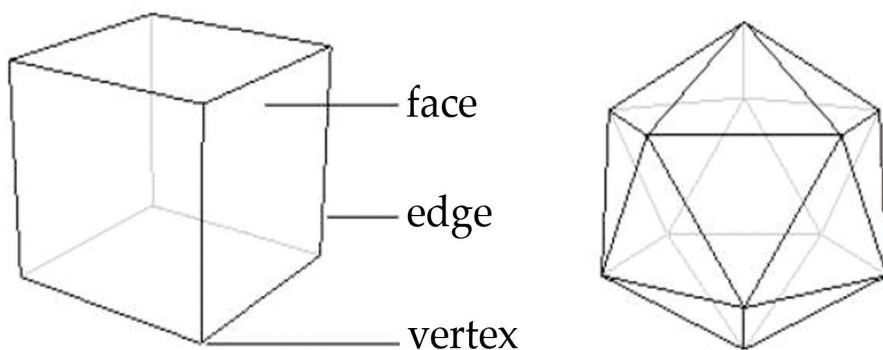


Figure 3: The familiar cube on the left and the icosahedron on the right. A polyhedron consists of polygonal faces, their sides are known as edges, and the corners as vertices.

A polyhedron consists of just one piece. It cannot, for example, be made up of two (or more) basically separate parts joined by only an edge or a vertex. This means that neither of the following objects is a true polyhedron.

For most polyhedra, calculating the surface area involves finding the sum of the areas of all its faces. Remember to consider the different shapes of faces and their dimensions. It's helpful to break down complex polyhedra into simpler shapes when calculating surface area.

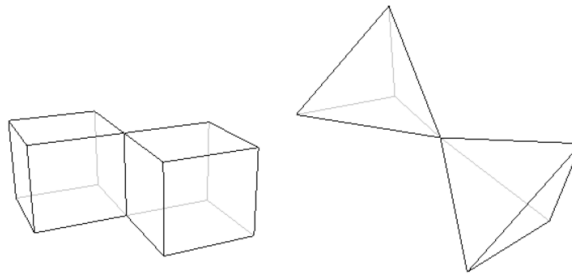


Figure 4: These objects are not polyhedra because they are made up of two separate parts meeting only in an edge (on the left) or a vertex (on the right).

Euler's formula of Polyhedron

A polyhedron, for example the cube or the icosahedron above, count the number of vertices it has, and call this number V . The cube, for example, has 8 vertices, so $V = 8$. Next, count the number of edges the polyhedron has, and call this number E . The cube has 12 edges, so in the case of the cube $E = 12$. Finally, count the number of faces and call it F . In the case of the cube, $F = 6$. Now Euler's formula tells us that

$$V - E + F = 2;$$

or, in words: the number of vertices, minus the number of edges, plus the number of faces, is equal to two.

In the case of the cube, we've already seen that $V = 8$, $E = 12$ and $F = 6$. So,

$$V - E + F = 8 - 12 + 6 = 14 - 12 = 2$$

which is what Euler's formula tells us it should be. If we now look at the icosahedron, we find that $V = 12$, $E = 30$ and $F = 20$. Now,

$$V - E + F = 12 - 30 + 20 = 32 - 30 = 2,$$

as we expected.

Euler's formula is true for the cube and the icosahedron. It turns out, rather beautifully, that it is true for pretty much every polyhedron. The only polyhedra for which it doesn't work are those that have holes running through them like the one shown in the figure below.

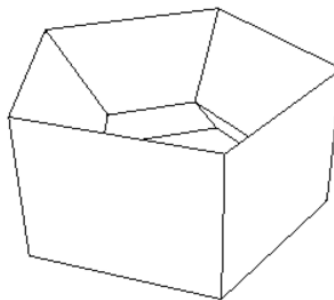


Figure 5: This polyhedron has a hole running through it. Euler's formula does not hold in this case.

These polyhedra are called non-simple, in contrast to the ones that don't have holes, which are called simple. Non-simple polyhedra might not be the first to spring to mind, but there are many of them out there, and we can't get away from the fact that Euler's Formula doesn't work for any of them. However, even this awkward fact has become part of a whole new theory about space and shape.

Power of Euler's Formula

Whenever mathematicians hit on an invariant feature, a property that is true for a whole class of objects, they know that they're onto something good. They use it to investigate what properties an individual object can have and to identify properties that all of them must have. Euler's formula can tell us, for example, that there is no simple polyhedron with exactly seven edges. You don't have to sit down with cardboard, scissors and glue to find this out the formula is all you need. The argument showing that there is no seven-edged polyhedron is quite simple, so have a look at it if you're interested.

Using Euler's formula in a similar way we can discover that there is no simple polyhedron with ten faces and seventeen vertices. The prism shown below, which has an octagon as its base, does have ten faces, but the number of vertices here is sixteen. The pyramid, which has a 9-sided base, also has ten faces, but has ten vertices. But Euler's formula tells us that no simple polyhedron has exactly ten faces and seventeen vertices.

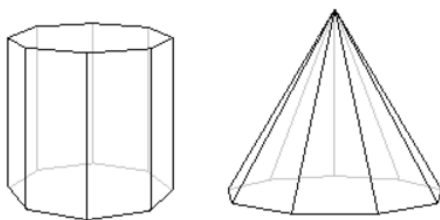


Figure 6: Both these polyhedra have ten faces, but neither has seventeen vertices.

It's considerations like these that lead us to what's probably the most beautiful discovery of all. It involves the Platonic Solids, a well-known class of polyhedra named after the ancient Greek philosopher Plato, in whose writings they first appeared.



Figure 7: The Platonic solids. From left to right we have the tetrahedron with four faces, the cube with six faces, the octahedron with eight faces, the dodecahedron with twelve faces, and the icosahedron with twenty faces.

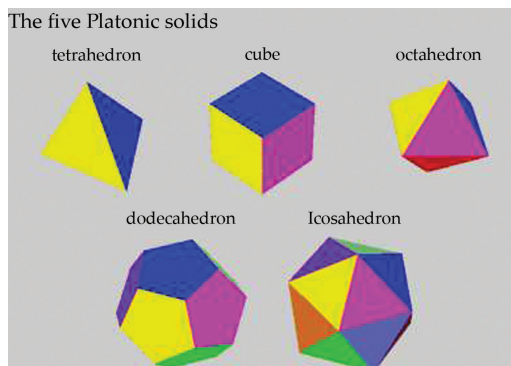
Although their symmetric elegance is immediately apparent when you look at the examples above, it's not actually that easy to pin it down in words. It turns out that it is described by two features. The first is that Platonic solids have no spikes or dips in them, so their shape is nice and rounded. In other words, this means that whenever you choose two points in a **Platonic solid** and draw a straight line between them, this piece of straight line will be completely contained within the solid. A Platonic solid is what is called convex. The second feature, called regularity, is that all the solid's faces are regular polygons with exactly the same number of sides, and that the same number of edges come out of each vertex of the solid.

Platonic solid is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex.

The cube is regular, since all its faces are squares and exactly three edges come out of each vertex. You can verify for yourself that the tetrahedron, the octahedron, the icosahedron and the dodecahedron are also regular.

Platonic Solid

Platonic solid, any of the five geometric solids whose faces are all identical, regular polygons meeting at the same three-dimensional angles. Also known as the five regular polyhedra, they consist of the tetrahedron (or pyramid), cube, octahedron, dodecahedron, and icosahedron. Pythagoras probably knew the tetrahedron, cube, and dodecahedron. According to Euclid, the octahedron and icosahedron were first discussed by the Athenian mathematician Theaetetus. However, the entire group of regular polyhedra owes its popular name to the great Athenian philosopher Plato, who in his dialogue *Timaeus* associated them with the four basic elements—fire, air, water, and earth—that he supposed to form all matter through their combinations. Plato assigned the tetrahedron, with its sharp points and edges, to the element fire; the cube, with its four-square regularity, to earth; and the other solids concocted from triangles (the octahedron and the icosahedron) to air and water, respectively. The one remaining regular polyhedra, the dodecahedron, with 12 pentagonal faces, Plato assigned to the heavens with its 12 constellations. Because of Plato's systematic development of a theory of the universe based on the five regular polyhedra, they became known as the Platonic solids.



Tetrahedron

A tetrahedron is a type of polyhedron which has four faces, making it the smallest possible type of polyhedron. This geometric figure is the basis for a wide variety of geometry problems, and examples of tetrahedra can be seen in architecture, the arts, and even daily life. In fact, chances are very good that there's a tetrahedron in your vicinity.

To understand the tetrahedron, it is unfortunately necessary to discuss a few key terms in geometry. A polygon is a flat or “planar” shape created with a series of connecting line segments: a triangle, for example, is a polygon. A polyhedron is a three-dimensional object composed of multiple polygons which meet to form straight edges. A well-known example of a polyhedron is a cube, a six-sided polyhedron. If the edges are curved, as in the case of a cylinder, the shape is no longer a polyhedron.

In the case of a tetrahedron, the polygons are all triangles by default, because in order to create a three-dimensional object with four polygons, each polygon must have three sides to connect with the other three polygons. The triangles can come in a variety of styles: when equilateral triangles are used, a tetrahedron is known as a “regular tetrahedron.” Tetrahedra are also sometimes called triangular pyramids, because they include a flat base and an apex.

There are lots of ways to play with this shape in mathematics. Triangles themselves are very interesting shapes from a mathematical standpoint, so an assortment of triangles is all the more interesting. Tetrahedra can also be joined together to create numerous other polyhedra, especially in the case of regular tetrahedra.

The tetrahedron is an example of a convex polyhedron. This means that if you randomly select any two points on the tetrahedron and connect them with a line, the line will pass through the tetrahedron, and not stray outside of it. By contrast, in a non-convex polyhedron, the line would at some point travel outside the polyhedron. Generally, the more faces a polyhedron has, the harder it is to make it convex, and at a certain point, it must become non-convex to accommodate all the faces.

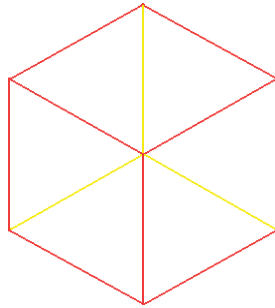
Some architects like to use this shape to add visual interest to their designs. Some cultures have also historically attached religious significance to this shape, or to collections of tetrahedra. The star tetrahedron, for example, is a polyhedron created

by merging two tetrahedra which face in opposite directions, creating an eight-pointed star.

Cube or Hexahedron

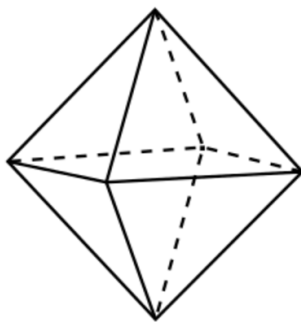
The **hexahedron**, otherwise called a cube. It is a platonic solid made from squares and has three sides meeting at each corner. This particular solid is made from a total of six squares, and you can see below that at each of the eight corners there are three square sides coming together. The cube, or hexahedron, is a popular solid in the world. You see it in ice cubes and also in the traditional six-sided playing dice.

Hexahedron is any polyhedron with six faces. A cube, for example, is a regular hexahedron with all its faces square, and three squares around each vertex.



Octahedron

The octahedron is one of five solids in the set of Platonic solids. These solids get their name from Plato, not because he discovered them but because he mentioned them often in his writings. He assigned each solid to be a representation of the natural elements. The octahedron was the symbol for air.



Parts of an Octahedron

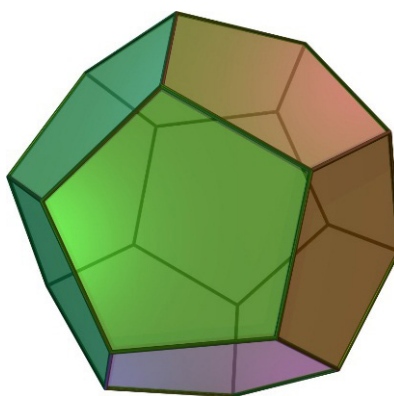
The octahedron is a three dimensional object and is composed of one and two dimensional parts. These parts have special names.

- *Base:* The base of an octahedron is a square. If you picture an octahedron as two congruent square pyramids that have their bottoms touching, then the base of the octahedron is the square between the two pyramids.
- *Face:* An octahedron has eight faces, which are all in the shape of equilateral triangles. These eight faces are where the solid gets its name. Octa means eight. These faces form the surface area of the octahedron. The square that is the base of the octahedron is not part of the surface area; therefore, the base is not also a face.
- *Edge:* When two faces touch, the line segment that is formed is called an edge. An octahedron has 12 edges.
- *Vertex:* When two edges intersect they form a vertex (plural being vertices). The octahedron has six vertices. Each vertex is formed when four edges intersect.

Dodecahedron

Dodecahedrons are any polyhedron with twelve flat faces. The most familiar dodecahedron is the regular dodecahedron, which is a Platonic solid.

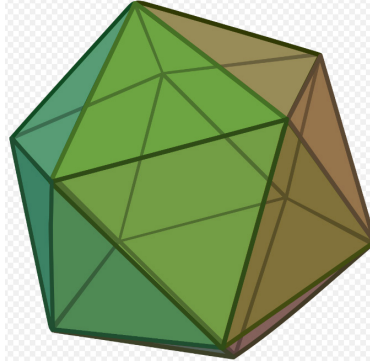
Another platonic solid is the dodecahedron, which is a platonic solid that has pentagons for its sides and that has three pentagon sides meeting at every corner. This solid has a total of twelve pentagons and twenty corners. In the real world, you will see **dodecahedrons** as fancy playing dice in some games. Usually, these are role-playing games that require a twelve-sided die.



Icosahedron

The last of the platonic solids is the icosahedron, a platonic solid with triangles for sides and that has five triangle sides meeting at every corner. This particular solid has a total of

twenty triangles and twelve corners. This platonic solid has the largest number of sides out of all the others.

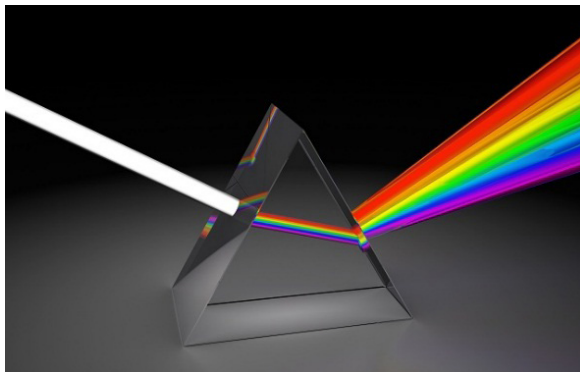


Prism

Prism refers to a certain three-dimensional shape: a polyhedron or star polygon connected to an identical shape by rectangular sides. A cube or rectangular box is one example of this type of prism, but instead of having a rectangle on the top and bottom, as in this example, a geometrical prism may also have a triangle, a six-pointed star, or any other many-sided shape. In optics, a prism is a transparent object with smooth, flat sides that refracts light. Many optical prisms are also geometrical prisms, often triangular prisms, but they may also be other shapes. An optical prism is commonly made of glass or clear plastic, but any material may be used as long as it is transparent to the light source that is to be refracted.

A prism is the solid shape you get if you move a flat polygon through space. A prism has five planes: a top, a bottom, and three sides. The most common kind of prism is a rectangular prism, made by moving a quadrilateral through space: a cube is a common example of a rectangular prism. The next most common kind of prism is a triangular prism, made by moving a triangle through space.

To figure out the surface area of a prism, calculate the surface area of each side and add them together. On a rectangular prism, each rectangular side is congruent (the same size) with the another one parallel to it. On a triangular prism, three of the sides are rectangles, and two of them are triangles. At least two of the rectangles are congruent, and the two triangles are also congruent.

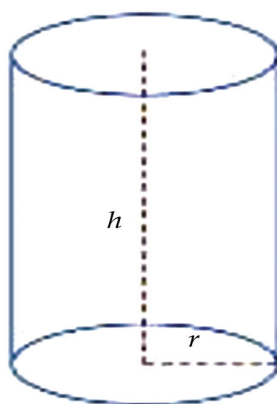


To figure out the volume of a triangular prism, think of the prism as a rectangular solid sliced in half. So you multiply the length by the height by the thickness, and then divide the result in half. One useful thing that is in the shape of a triangular prism is a camping tent. Why might you need to know the volume of a tent? The attic of a house might also be in the shape of a triangular prism.

Cylinder

A Cylinder is a solid with circular base. It can be of the shape of a pillar, a rubber tube, the trunk of a tree, etc. The term circular cylinder is used to describe a right circular cylinder. A circular cylinder having perpendicular base and height is known as a right circular cylinder.

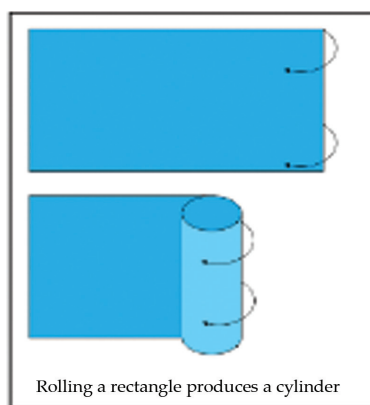
The diagram given here is of a right circular cylinder:



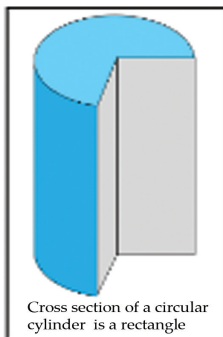
In the above diagram, h = height of the circular cylinder

r = radius of base of circular cylinder

We can also say that a circular cylinder is a rolled form of a rectangle. The area of the rectangle that forms the shape of a cylinder is called the curved surface area of a cylinder, since the rectangle forms the curved surface of the cylinder.



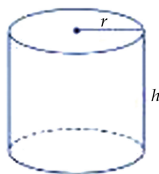
When we cut a cylinder vertically, the cross section obtained is a rectangle. In other words, when a rectangle is revolved around one of its sides, a cylinder is obtained.



When we add the area of the top and bottom circles of a cylinder to its curved surface area, we get the total surface area of a cylinder.

The solid enclosed by the surface and by two planes perpendicular to the axis is called as a cylinder. A cylinder is a closed three dimensional solid which is made up of two parallel circular bases connected by a curved surface.

Cylinder Shape:



In this figure: r is the radius of a cylinder and h is the height of a cylinder which is perpendicular distance between the bases.

Cylinder Volume

The amount of space occupied by the cylinder is its volume. The volume of a cylinder is the area of its circular base times its height. Volume of cylinder is measured in cubic units.

Cylinder Volume Formula:

Formula for volume of a cylinder = $\pi r^2 h$

where, r : Radius of a cylinder

h : Height of a cylinder

and π is equal to $\frac{22}{7}$ or 3.14

Curved Surface Area of a Cylinder

A cylinder is a solid with two congruent, parallel bases. These bases are connected by a rounded surface. A right circular cylinder with circular bases that are directly above and below each other.

The volume of a cylinder is obtained by multiplying the area of the base by height. The volume of a cylinder is given by $V = \pi r^2 h$ where v is volume, r is the radius and

h is the height.

The curved surface area of a cylinder is given by the following formula,

$$\text{Curved Surface Area} = 2 * \pi * r * h$$

where, r = radius of the base circles of cylinder

h = height of the cylinder

Cylinder Surface Area

Surface area of a cylinder is the sum of curved surface area and area of each base of the cylinder.

Total Surface Area = Curved surface area + Area of top + Area of bottom

$$= 2 \pi r h + \pi r^2 + \pi r^2 (\because \text{Top and bottom of a cylinder are circular})$$

$$= 2\pi r(r + h)$$

Cylinder Formula

Cylinder area = $2\pi r(r + h)$

where, r = Radius of a cylinder

h = Height of a cylinder

Pyramid

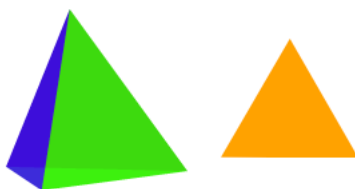
A pyramid is a 3-dimensional geometric shape formed by connecting all the corners of a polygon to a central apex. Each base edge and apex form a triangle, called a lateral face. It is a conic solid with polygonal base. A pyramid with an n-sided base has n + 1 vertices, n + 1 faces, and 2n edges. All pyramids are self-dual.

Types of Pyramids

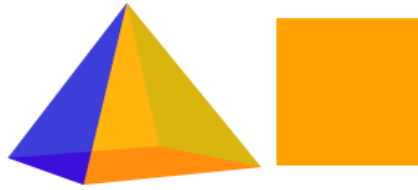
A pyramid is a 3-dimensional shape whose base is a polygon. Each corner of a polygon is attached to a singular apex, which gives the pyramid its distinctive shape. Each base edge and the apex form a triangle.

There are many types of pyramids. Most often, they are named after the type of base they have. Let's look at some common types of pyramids below.

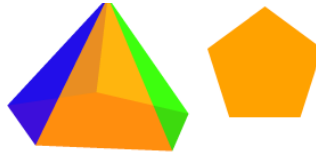
Triangular pyramid has a triangle as its base:



Square pyramid has a square as its base:



Pentagonal pyramid has a pentagon as its base:



Right pyramid: the apex of the pyramid is directly above the center of its base:

Oblique pyramid: the apex of the pyramid is not directly above the center of its base:

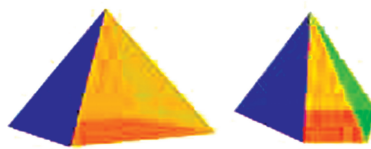


Right Pyramid

Oblique Pyramid

Regular pyramid: the base of this pyramid is a regular polygon:

Irregular pyramid: this type of pyramid has an irregular polygon as its base:



Regular Pyramid

Irregular Pyramid

Pyramid Formulas

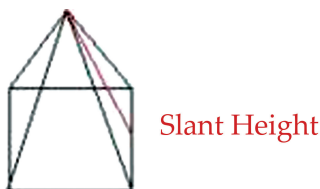
There are formulas that can be used to find both the surface area and the volume of a pyramid. The surface area of a pyramid is the total area of all the surfaces that the pyramid has. To that end, the formula for finding the surface area when all of the side faces are the same is:

$$SA = (\text{base area}) + (1/2) * (\text{perimeter}) * (\text{slant height})$$

The base area is the area of the base and can be determined based on what figure the base is. For example, if the base of the pyramid is a square, the formula for finding the area is:

$$A = s^2$$

The perimeter is the distance around the base of the pyramid. The slant height is the diagonal height from the center of one of the base edges to the apex.



If the pyramid has side faces that differ from each other (like in the case of an irregular pyramid), then the surface area equation is:

$$SA = (\text{base height}) + (\text{lateral area})$$

In this case, you must take each side of the pyramid separately (including the base), find the areas, and then just add them together.

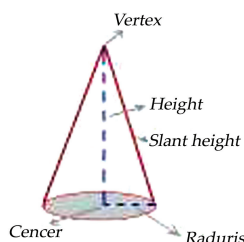
The volume of a pyramid can be found using this formula:

$$V = (1/3) * (\text{base area}) * (\text{height})$$

Cone

Cone is a three-dimensional geometric shape that tapers smoothly from a flat base to a point called the apex or vertex.

Two dimensional shapes possess length and breadth, but no height. While, the three-dimensional shapes have all three of them. General three-dimensional or solid shapes are cube, sphere, cuboid, prism, cylinder, pyramid, cone etc. A **cone** is a three dimensional geometric shape. For example, If one end of a line is twisted about a second set line while keeping line's other end fixed, we get a cone. The point about which the line is curved is known as the vertex and the foundation or base of the cone is a circle. The vertex is directly above the centre of the bottom.

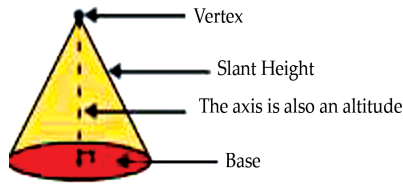


There are two types of cones

- right circular cone

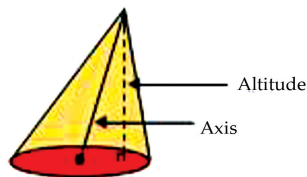
- oblique cone

A right circular cone is a kind of cone in which the axis passing through the center of cone is inclined exactly at right angles to the plane of base. In other words, axis and height of a right cone are same. It is demonstrated in the diagram as under.



Right Cone

On the other hand, an oblique cone does not have its axis perpendicular to the plane of its base. So for oblique cone, the height and axis are different as shown in the image.



Oblique Cone

Properties of a Cone

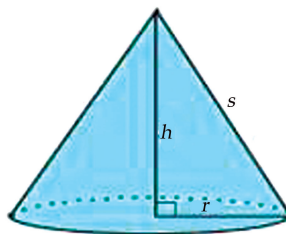
There are a number of properties of a cone.

Some of the properties of a cone are:

- Volume of a Cone
- Face Surface Area of a Cone
- Total Surface Area of a Cone

Cone Formula

For a right circular cone of radius r , height h and slant height s , we have



Area of the curved surface or lateral surface of the cone = $\pi r s$ sq. units

Total surface area of the cone = $\pi r(s + r)$ sq. units

Volume of the cone = $\frac{1}{3} \pi r^2 h$ cubic units.

Cone Volume

The volume of a cone is exactly one third the volume of a cylinder of the same height.

Volume of a cone = $\frac{1}{3}$ (Volume of a cylinder)

$$= \frac{1}{3} (\pi r^2 h) [\because \text{Volume of a cylinder} = \pi r^2 h]$$

Cone Volume Formula

The volume of a cone formula is as follows,

$$V = \frac{\pi}{3} r^2 h$$

Where, r - radius

h - height of a cone.

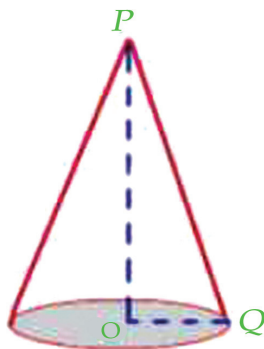
Area of Cone

The surface area of the cone depend upon the radius of base and slant height of the cone. The surface area of a cone whose radius of the base r , the slant height s , and the area of the lateral surface S is,

$$\text{Cone surface area (S)} = \pi r(r + s)$$

Right Circular Cone

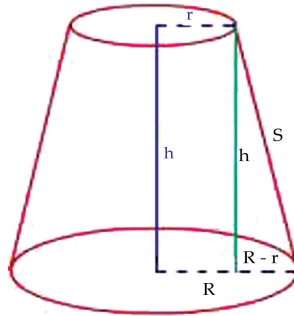
If a right triangle is revolved about one of the sides containing the right angle, then the solid thus generated, is called a right circular cone.



In this figure: Right angled triangle POQ is revolved about PO, it generates a right circular cone.

Truncated Cone

If we cut a right circular cone with a plane parallel to the base of the cone, then the solid shape formed between the base of the cone and plane is called the truncated cone.



Formulas for the truncated cone: For a truncated cone of radius r and R , height h and slant height s .

$$\text{Slant Height, } s = \sqrt{(R - r)^2 + h^2}$$

$$\text{Lateral Surface Area, LSA} = \pi(R + r) S \text{ sq. units}$$

$$\text{Total Surface Area, SA} = \pi[(R + r)s + R^2 + r^2] \text{ square units}$$

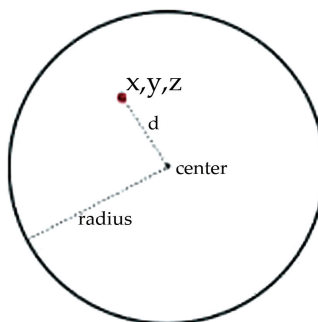
$$\text{Volume, } V = \frac{1}{3} \pi h[Rr + R^2 + r^2] \text{ cubic units}$$

Sphere

A sphere is a geometrical figure that is perfectly round, 3-dimensional and circular - like a ball. Geometrically, a sphere is defined as the set of all points equidistant from a single point in space. The distance from an outer point to the center of the sphere is the radius(r) and the maximum straight distance from one side of the sphere to another is the diameter (d).

A hemisphere is what you would call half a sphere, if you were to split a sphere down the middle.

A great circle of the sphere is a circle that has the same radius and center as the sphere itself.



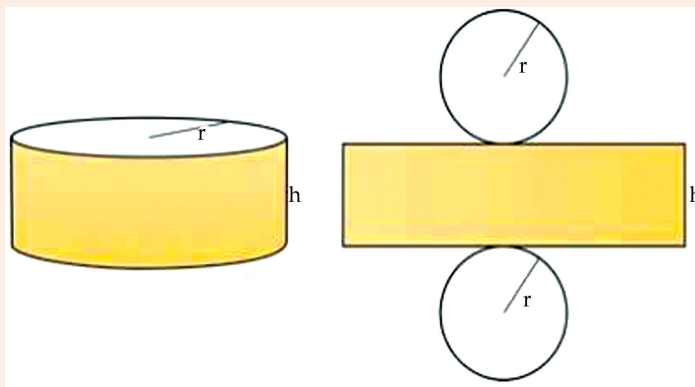
Properties

A sphere is the geometrical figure that occupies the biggest space, but has the smallest surface area. In other words, when something needs to be as small as possible but still have a large volume, it takes the shape of a sphere. That is why a balloon is round when you blow it up. It wants to hold as much air as possible with the smallest amount of surface. This occurs quite often in nature - common examples include bubbles and water drops.

The planet Earth is called a spheroid because it is extremely close to being a sphere, but is not perfectly round. It is elongated a bit at the North and South poles.

CONCEPT OF HOW TO FIND THE SURFACE AREA OF A CYLINDER?

To estimate the surface area of a cylinder, you need to visualize it as a net. It is as if you open the cylinder just like a carton box and flatten it out. Use your imagination! And what will you get? The answer is that a right circular cylinder consists of two circles and one rectangle, as you can see in the figure below.



Therefore, the base surface area of a cylinder equals two times the area of a circle with the radius r , and the lateral surface area of a cylinder is the area of a rectangle. The first side of this rectangle is the height of the cylinder h , and the second is the circumference of the base equal to $2 \times \pi \times r$, according to the circumference calculator.

SUMMARY

- A perimeter is a path that surrounds a two-dimensional shape. The term may be used either for the path or its length it can be thought of as the length of the outline of a shape. The perimeter of a circle or ellipse is called its circumference.
- The area of a polygon is the number of square units inside that polygon. Area is 2-dimensional like a carpet or an area rug. A triangle is a three-sided polygon.
- A polyhedron is a solid object whose surface is made up of a number of flat faces which themselves are bordered by straight lines.
- Platonic solid, any of the five geometric solids whose faces are all identical, regular polygons meeting at the same three-dimensional angles. Also known as the five regular polyhedra, they consist of the tetrahedron (or pyramid), cube, octahedron, dodecahedron, and icosahedron.
- In the case of a tetrahedron, the polygons are all triangles by default, because in order to create a three-dimensional object with four polygons, each polygon must have three sides to connect with the other three polygons.
- Another platonic solid is the dodecahedron, which is a platonic solid that has pentagons for its sides and that has three pentagon sides meeting at every corner.
- A Cylinder is a solid with circular base. It can be of the shape of a pillar, a rubber tube, the trunk of a tree, etc. The term circular cylinder is used to describe a right circular cylinder. A circular cylinder having perpendicular base and height is known as a right circular cylinder.
- A right circular cone is a kind of cone in which the axis passing through the center of cone is inclined exactly at right angles to the plane of base.
- A sphere is a geometrical figure that is perfectly round, 3-dimensional and circular - like a ball. Geometrically, a sphere is defined as the set of all points equidistant from a single point in space. The distance from an outer point to the center of the sphere is the radius(r) and the maximum straight distance from one side of the sphere to another is the diameter (d).

MULTIPLE CHOICE QUESTIONS

1. **By converting 5.6m^2 into cm^2 , answer will be**
 - a. 0.0056cm^2
 - b. 0.56cm^2
 - c. 56000cm^2
 - d. 560cm^2
2. **A rectangular field is 40m long and 30m wide. perimeter of rectangular field is**
 - a. 200m^2
 - b. 180m^2
 - c. 160m^2
 - d. 140m^2
3. **By converting 0.96km^2 into m^2 (meter square), answer will be**
 - a. 9600m^2
 - b. 960m^2
 - c. 0.96m^2
 - d. 960000m^2
4. **Type of quadrilateral which has one pair of parallel sides is called**
 - a. triangle
 - b. semi-circle
 - c. parallelogram
 - d. trapezium
5. **If width of rectangle is 10cm less than its length and its perimeter is 50cm then width of rectangle is**
 - a. 58cm^2
 - b. 64cm^2
 - c. 15cm^2
 - d. 30cm^2
6. **What is the perimeter of a rectangle with a length of 10 units and a width of 6 units?**
 - a. 16 units
 - b. 36 units
 - c. 32 units
 - d. 22 units
7. **If the radius of a circle is 5 cm, what is its circumference (rounded to the nearest whole number)?**
 - a. 15 cm

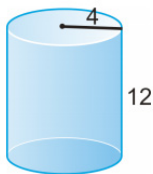
- b. 31 cm
 - c. 25 cm
 - d. 16 cm
8. **A triangular garden has sides measuring 8 meters, 15 meters, and 17 meters. What is its area?**
- a. 64 square meters
 - b. 60 square meters
 - c. 56 square meters
 - d. 72 square meters
9. **The base of a trapezoid is 12 cm, the top is 8 cm, and the height is 5 cm. What is its area?**
- a. 45 square cm
 - b. 50 square cm
 - c. 40 square cm
 - d. 48 square cm
10. **A regular hexagon has a side length of 7 units. What is its perimeter?**
- a. 21 units
 - b. 35 units
 - c. 42 units
 - d. 49 units

Answer

- | | | | | |
|--------|--------|--------|--------|---------|
| 1. (c) | 2. (d) | 3. (d) | 4. (d) | 5. (c) |
| 6. (b) | 7. (b) | 8. (a) | 9. (c) | 10. (b) |

REVIEW QUESTIONS

1. Describe the concept of perimeter and area.
2. Find the surface area of the cylinder.



3. Define the surface area and volume.
4. Discuss the power of Euler's formula.
5. How many types of pyramids in geometry? Explain it.
6. Draw the net of a square pyramid.
7. What shape are the bases of this prism? What are their areas?

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Core Concepts in Mathematics: Geometry

2nd Edition

In the early seventeenth century, the groundbreaking development of symbolic algebra transformed mathematics, clearing the path for a more intricate period of investigation. The tool enabled mathematicians to approach problems in a more flexible and powerful way, free from the constraints of specific numerical values.

Symbolic algebra's might resided in its capacity to perform calculations and manipulations using variables and symbols, independent of a numerical context. Moving away from concrete calculations, mathematicians could explore the underlying structures and patterns within mathematical problems, resulting in a deeper grasp of the subject. With symbolic algebra's introduction, mathematical ideas and unsolved issues became more accessible.

The shift towards analytical techniques in mathematics became more pronounced during the 19th century. Geometric methods were outdone by their more reliable and systematic counterparts, rooted in equations and functions. Subsequently, geometry's emphasis as a tool for mathematical concept communication waned in numerous textbooks. Unforeseen consequences of this shift affected how mathematics was taught and absorbed. The absence of geometry, incorporating visual and spatial elements, contributed to a lessened grasp of the subject. The difficulty many students faced in grasping abstract concepts depend on the absence of geometric insights in their analytical operation.

Although, our daily lives are linked to spatial, geometric, and quantitative concepts. The universe's diverse forms are highlighted by the Earth and its celestial neighbors. Each celestial body has its distinct shape and size, combining to create the elaborate universe. Spatial reasoning is a crucial element in unlocking the world's complex forms. Spatial relationships, positions, sizes, and shapes are intricately connected and require a deeper understanding within a specific context.

The importance of geometry in cultivating this spatial aptitude is fundamental. Geometry offers a system for grasping and navigating our surroundings by furnishing a basis for measurements, angle calculations, line visualization, surface comprehension, and solid structure appreciation. Geometric concepts offer us the power to calculate distances, build structures, and resolve practical issues with precision and speed. The visual and spatial aspects of geometry help cultivate creativity and intuition, serving mathematical and scientific fields well.

"Core Concepts in Geometry" 2nd edition builds a robust foundation by emphasizing critical elements. The latest edition contains revised chapters with additional topics. The limitations of students' geometric knowledge are largely attributable to deficiencies within educational institutions. Despite the availability of numerous tools for creating and manipulating geometric shapes, many struggle to fully understand the wealth of information they contain. Comprising diverse topics, this book offers readers a launching point into central concepts of rapidly transforming geometric spheres.

Diego De Leo obtained his PhD in Algebraic topology and Homotopy theory. His research areas include differential geometry, algebraic geometry, computational geometry, algebraic topology, discrete geometry. He has written several articles and papers on symplectic, Riemannian, and complex manifolds, with applications to and from combinatorics, classical and quantum physics, ordinary and partial differential equations, and representation theory.