



Core Concepts in

Mathematics: Trigonometry

Duried Alwazeer

2ND EDITION

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CORE CONCEPTS IN MATHEMATICS:
TRIGONOMETRY

Duried Alwazeer

2nd Edition



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Preface

Trigonometry deals with the relationships between angles and sides of triangles, plays a crucial role in various fields. From engineering to physics and even music, its applications are extensive. In the realm of engineering, trigonometry is indispensable. Engineers rely on this subject to design structures such as bridges and buildings. By using trigonometric functions like sine, cosine, and tangent, they can accurately calculate angles and distances. This enables them to ensure stability, efficiency, and safety in their designs. Moreover, trigonometry finds widespread use in physics. It helps physicists analyze forces acting on objects by breaking them down into components along different axes. The knowledge of trigonometric principles assists them in understanding complex motions involving rotation or oscillation accurately. Furthermore tangents can be found within sound waves while studying acoustics—another application area for Trigonometry importance lies here too inside audio production becoming significant due to frequency response calculations ensuring clean reception composition. Another interesting field where trigonometry shines is astronomy! Astronomers utilize this mathematical tool when calculating celestial positions such as determining stellar parallax or measuring the distance between galaxies—and it's an essential aspect more generally throughout space research.

Core Concepts in Trigonometry (2nd Edition) presents students and self-learners with a lucid and comprehensible exploration of trigonometry's fundamental principles. It delves into the examination of angles and dimensions within triangles. Although this may appear straightforward, trigonometry plays an integral role in modern engineering, design, architecture, and various other disciplines. Moreover, it consistently furnishes students and educators with well-structured explanations of mathematical concepts. The application of trigonometry in geometry involves the computation of side lengths and angles primarily within right-angled triangles. Geothermal applications serve as the most prevalent utilization of trigonometry whereby information about a triangle can unveil precise measurements pertaining to its sides and angles. This book aims to cultivate your ability to think mathematically efficiently. Through its guidance, you can transcend mere rote memorization or imitation of examples—instead fostering genuine problem-solving skills that will endure throughout time.

INTRODUCTION TO TRIGONOMETRIC FUNCTIONS



Trigonometry comes from the two roots, trigonon (or “triangle”) and metria (or “measure”). The study of trigonometry is thus the study of measurements of triangles. What can we measure in a triangle? The first objects that come to mind may be the lengths of the sides, the angles of the triangle, or the area contained in the triangle. We first explore trigonometric functions that connect the measures of angles with the lengths of the sides.

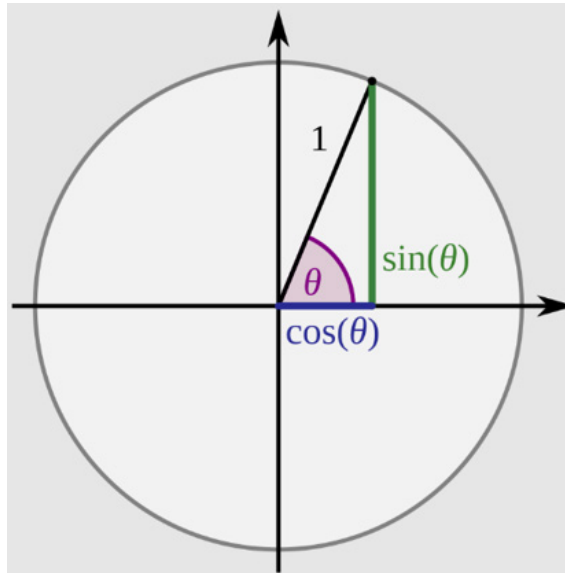
After studying this chapter, you will understand the core concepts of:

- Principles of Trigonometry
- Trigonometric Functions

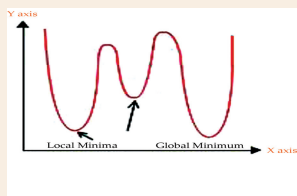
INTRODUCTION

Trigonometric functions are also known as circular functions can be simply defined as the functions of an angle of a triangle. It means that the relationship between the angles and sides of a triangle are given by these trig functions. The basic trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. There are a number of trigonometric formulas and identities that denotes the relation between the functions and help to find the angles of the triangle.

The trigonometric functions most widely used in modern mathematics are the sine, the cosine, and the tangent. Their reciprocals are respectively the cosecant, the secant, and the cotangent, which are less used. Each of these six trigonometric functions has a corresponding inverse function, and an analog among the hyperbolic functions.



The oldest definitions of trigonometric functions, related to right-angle triangles, define them only for acute angles. To extend the sine and cosine functions to functions whose domain is the whole real line, geometrical definitions using the standard unit circle (i.e., a circle with radius 1 unit) are often used; then the domain of the other functions is the real line with some isolated points removed. Modern definitions express trigonometric functions as infinite series or as solutions of differential equations. This allows extending the domain of sine and cosine functions to the whole complex plane, and the domain of the other trigonometric functions to the complex plane with some isolated points removed.



CORE CONCEPT: PRINCIPLES OF TRIGONOMETRY

Trigonometry is a specialist branch of geometry that deals with the study of triangles. It is sometimes informally referred to as “trig.” In trigonometry, mathematicians study the relationships between the sides and angles of triangles. Right triangles, which are triangles with one angle of 90 degrees, are a key area of study in this area of mathematics.

Trigonometry, the branch of mathematics concerned with specific functions of angles and their application to calculations. There are six functions of an angle commonly used in trigonometry. Their names and abbreviations are sine (sin), cosine (cos), tangent (tan), cotangent (cot), secant (sec), and cosecant (csc). These six trigonometric functions in relation to a right triangle are displayed in the figure. For example, the triangle contains an angle A , and the ratio of the side opposite to A and the side opposite to the right angle (the hypotenuse) is called the sine of A , or $\sin A$; the other trigonometry functions are defined similarly. These functions are properties of the angle A independent of the size of the triangle, and calculated values were tabulated for many angles before computers made trigonometry tables obsolete. Trigonometric functions are used in obtaining unknown angles and distances from known or measured angles in geometric figures.

Trigonometry developed from a need to compute angles and distances in such fields as astronomy, mapmaking, surveying, and artillery range finding. Problems involving angles and distances in one plane are covered in plane trigonometry. Applications to similar problems in more than one plane of three-dimensional space are considered in spherical trigonometry.

Geometry is a branch of mathematics concerned with questions of shape, size, relative position of figures, and the properties of space.

Basics of Trigonometric Functions

A somewhat more general concept of angle is required for trigonometry than for **geometry**. An angle A with vertex at V , the initial side of which is VP and the terminal side of which is VQ , is indicated in the figure by the solid circular arc. This angle is generated by the continuous counter clockwise

rotation of a line segment about the point V from the position VP to the position VQ . A second angle A' with the same initial and terminal sides, indicated in the figure by the broken circular arc, is generated by the clockwise rotation of the line segment from the position VP to the position VQ . Angles are considered positive when generated by counter clockwise rotations, negative when generated by clockwise rotations. The positive angle A and the negative angle A' in the figure are generated by less than one complete rotation of the line segment about the point V . All other positive and negative angles with the same initial and terminal sides are obtained by rotating the line segment one or more complete turns before coming to rest at VQ .

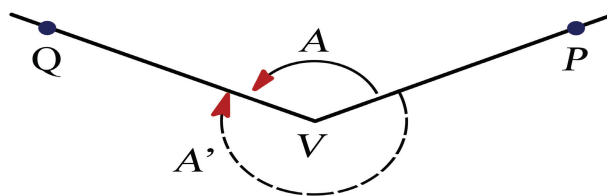


Figure 1: This figure shows a positive general angle A , as well as a negative general angle A' .

Numerical values can be assigned to angles by selecting a unit of measure. The most common units are the degree and the radian. There are 360° in a complete revolution, with each degree further divided into $60'$ (minutes) and each minute divided into $60''$ (seconds). In theoretical work, the radian is the most convenient unit. It is the angle at the centre of a circle that intercepts an arc equal in length to the radius; simply put, there are 2π radians in one complete revolution. From these definitions, it follows that $1^\circ = \pi/180$ radians.

Equal angles are angles with the same measure; i.e., they have the same sign and the same number of degrees. Any angle $-A$ has the same number of degrees as A but is of opposite sign. Its measure, therefore, is the negative of the measure of A . If two angles, A and B , have the initial sides VP and VQ and the terminal sides VQ and VR , respectively, then the angle $A + B$ has the initial and terminal sides VP and VR . The angle $A + B$ is called the sum of the angles A and B , and its relation to A and B when A is positive and B is positive or negative is illustrated in the figure.

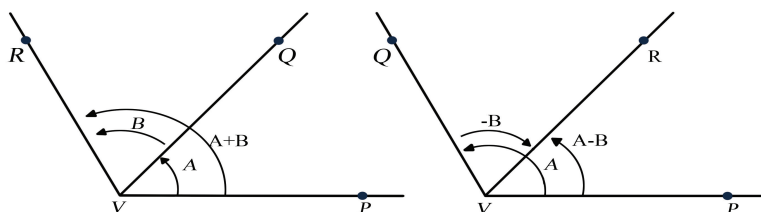


Figure 2: The figure indicates how to add a positive or negative angle (B) to a positive angle (A).

The sum $A + B$ is the angle the measure of which is the algebraic sum of the measures of A and B . The difference $A - B$ is the sum of A and $-B$. Thus, all angles

coterminal with angle A (i.e., with the same initial and terminal sides as angle A) are given by $A \pm 360n$, in which $360n$ is an angle of n complete revolutions. The angles $(180 - A)$ and $(90 - A)$ are the supplement and complement of angle A , respectively.

Trigonometric functions of an angle

To define trigonometric functions for any angle A , the angle is placed in position on a rectangular coordinate system with the vertex of A at the origin and the initial side of A along the positive x -axis; r (positive) is the distance from V to any point Q on the terminal side of A , and (x, y) are the rectangular coordinates of Q .

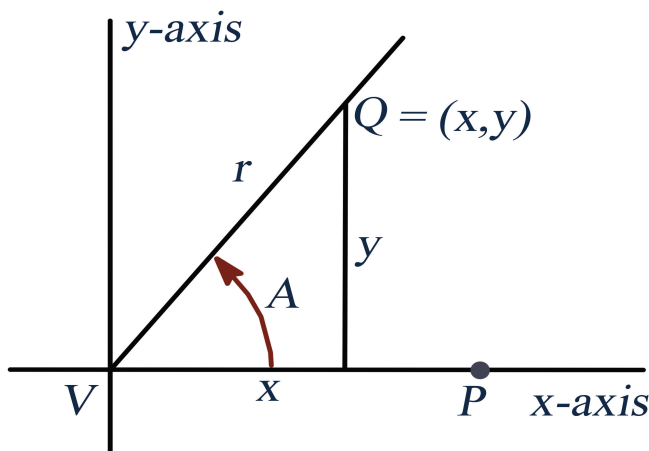


Figure 3: The figure shows an angle A in standard position, that is, with initial side on the x -axis.

The six functions of A are then defined by six ratios exactly as in the earlier case for the triangle given in the introduction. Because division by zero is not allowed, the tangent and secant are not defined for angles the terminal side of which falls on the y -axis, and the cotangent and cosecant are undefined for angles the terminal side of which falls on the x -axis. When the Pythagorean equality $x^2 + y^2 = r^2$ is divided in turn by r^2 , x^2 , and y^2 , the three squared relations relating cosine and sine, tangent and secant, cotangent and cosecant are obtained.

If the point Q on the terminal side of angle A in standard position has coordinates (x, y) , this point will have coordinates $(x, -y)$ when on the terminal side of $-A$ in standard position. From this fact and the definitions are obtained further identities for negative angles. These relations may also be stated briefly by saying that cosine and secant are even functions (symmetrical about the y -axis), while the other four are odd functions (symmetrical about the origin).

It is evident that a trigonometric function has the same value for all coterminal angles. When n is an integer, therefore, $\sin(A \pm 360n) = \sin A$; there are similar relations for the other five functions. These results may be expressed by saying that the trigonometric functions are periodic and have a period of 360° or 180° .

When Q on the terminal side of A in standard position has coordinates (x, y) , it has coordinates $(-y, x)$ and $(y, -x)$ on the terminal side of $A + 90$ and $A - 90$ in standard position, respectively. Consequently, six formulas equate a function of the complement of A to the corresponding cofunction of A .

Tables of natural functions

To be of practical use, the values of the trigonometric functions must be readily available for any given angle. Various trigonometric identities show that the values of the functions for all angles can readily be found from the values for angles from 0° to 45° . For this reason, it is sufficient to list in a table the values of sine, cosine, and tangent for all angles from 0° to 45° that are integral multiples of some convenient unit (commonly $1'$). Before computers rendered them obsolete in the late 20th century, such trigonometry tables were helpful to astronomers, surveyors, and engineers.

For angles that are not integral multiples of the unit, the values of the functions may be interpolated. Because the values of the functions are in general irrational numbers, they are entered in the table as decimals, rounded off at some convenient place. For most purposes, four or five decimal places are sufficient, and tables of this accuracy are common. Simple geometrical facts alone, however, suffice to determine the values of the trigonometric functions for the angles 0° , 30° , 45° , 60° , and 90° . These values are listed in a table for the sine, cosine, and tangent functions.

Plane trigonometry

In many applications of trigonometry, the essential problem is the solution of triangles. If enough sides and angles are known, the remaining sides and angles as well as the area can be calculated, and the triangle is then said to be solved. Triangles can be solved by the law of sines and the law of cosines. To secure symmetry in the writing of these laws, the angles of the triangle are lettered A , B , and C and the lengths of the sides opposite the angles are lettered a , b , and c , respectively.

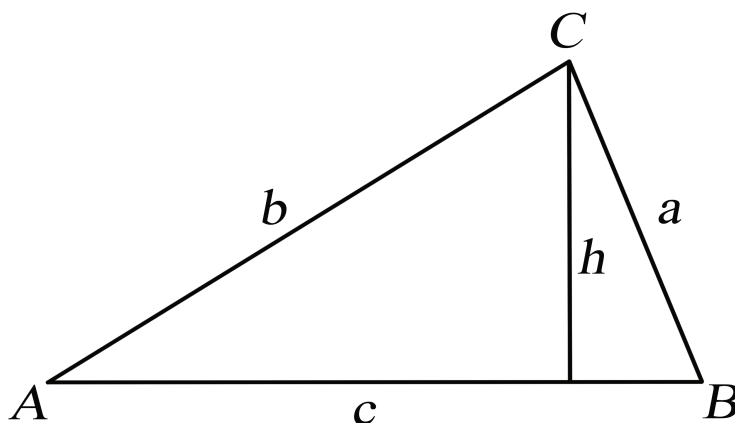


Figure 4: Standard lettering of a triangle In addition to the angles (A , B , C) and sides (a , b , c), one of the three heights of the triangle (h) is included by drawing the line segment from one

of the triangle's vertices (in this case C) that is perpendicular to the opposite side of the triangle.

The law of sines is expressed as an equality involving three sine functions while the law of cosines is an identification of the cosine with an algebraic expression formed from the lengths of sides opposite the corresponding angles. To solve a triangle, all the known values are substituted into equations expressing the laws of sines and cosines, and the equations are solved for the unknown quantities. For example, the law of sines is employed when two angles and a side are known or when two sides and an angle opposite one are known. Similarly, the law of cosines is appropriate when two sides and an included angle are known or three sides are known.

Texts on trigonometry derive other formulas for solving triangles and for checking the solution. Older textbooks frequently included formulas especially suited to logarithmic calculation. Newer textbooks, however, frequently include simple computer instructions for use with a symbolic mathematical program.

Spherical trigonometry

Spherical trigonometry involves the study of spherical triangles, which are formed by the intersection of three great circle arcs on the surface of a sphere. Spherical triangles were subject to intense study from antiquity because of their usefulness in navigation, cartography, and astronomy.

The angles of a spherical triangle are defined by the angle of intersection of the corresponding tangent lines to each vertex. The sum of the angles of a spherical triangle is always greater than the sum of the angles in a planar triangle (π radians, equivalent to two right angles). The amount by which each spherical triangle exceeds two right angles (in **radians**) is known as its spherical excess. The area of a spherical triangle is given by the product of its spherical excess E and the square of the radius r of the sphere it resides on—in symbols, Er^2 .

By connecting the vertices of a spherical triangle with the centre O of the sphere that it resides on, a special “angle” known as a trihedral angle is formed. The central angles (also known as dihedral angles) between each pair of line segments OA , OB , and OC are labeled α , β , and γ to correspond to the sides (arcs) of the spherical triangle labeled a , b , and c , respectively. Because a trigonometric function of a central angle and its

Radian is the SI unit for measuring angles, and is the standard unit of angular measure used in many areas of mathematics.

corresponding arc have the same value, spherical trigonometry formulas are given in terms of the spherical angles A , B , and C and, interchangeably, in terms of the arcs a , b , and c and the dihedral angles α , β , and γ . Furthermore, most formulas from plane trigonometry have an analogous representation in spherical trigonometry. For example, there is a spherical law of sines and a spherical law of cosines.

As was described for a plane triangle, the known values involving a spherical triangle are substituted in the analogous spherical trigonometry formulas, such as the laws of sines and cosines, and the resulting equations are then solved for the unknown quantities. Many other relations exist between the sides and angles of a spherical triangle. Worth mentioning are Napier's analogies (derivable from the spherical trigonometry half-angle or half-side formulas), which are particularly well suited for use with logarithmic tables.

Analytic trigonometry

Analytic trigonometry combines the use of a coordinate system, such as the Cartesian coordinate system used in analytic geometry, with algebraic manipulation of the various trigonometry functions to obtain formulas useful for scientific and engineering applications.

Real number is a value of a continuous quantity that can represent a distance along a line.

Trigonometric functions of a real variable x are defined by means of the trigonometric functions of an angle. For example, $\sin x$ in which x is a **real number** is defined to have the value of the sine of the angle containing x radians. Similar definitions are made for the other five trigonometric functions of the real variable x . These functions satisfy the previously noted trigonometric relations with A , B , 90° , and 360° replaced by x , y , $\pi/2$ radians, and 2π radians, respectively. The minimum period of $\tan x$ and $\cot x$ is π , and of the other four functions it is 2π .

In calculus it is shown that $\sin x$ and $\cos x$ are sums of power series. These series may be used to compute the sine and cosine of any angle. For example, to compute the sine of 10° , it is necessary to find the value of $\sin \pi/18$ because 10° is the angle containing $\pi/18$ radians. When $\pi/18$ is substituted in the series for $\sin x$, it is found that the first two terms give 0.17365, which is correct to five decimal places for the sine of 10° . By taking enough terms of the series, any number of decimal places can be correctly obtained. Tables of the functions may be used to sketch the graphs of the functions.

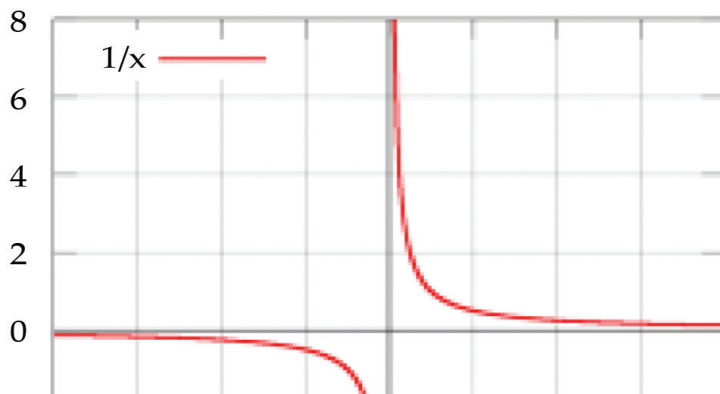


Figure 5: Graphs of some trigonometric functions.

Note that each of these functions is periodic. Thus, the sine and cosine functions repeat every 2π , and the tangent and cotangent functions repeat every π .

Each trigonometric function has an inverse function, that is, a function that “undoes” the original function. For example, the inverse function for the sine function is written $\arcsin$ or $\sin^{-1}$, thus $\sin^{-1}(\sin x) = \sin(\sin^{-1} x) = x$. The other trigonometric inverse functions are defined similarly.

Coordinates and transformation of coordinates

Polar coordinates

For problems involving directions from a fixed origin (or pole) O , it is often convenient to specify a point P by its polar coordinates (r, θ) , in which r is the distance OP and θ is the angle that the direction of r makes with a given initial line. The initial line may be identified with the x -axis of rectangular Cartesian coordinates, as shown in the figure. The point (r, θ) is the same as $(r, \theta + 2n\pi)$ for any integer n . It is sometimes desirable to allow r to be negative, so that (r, θ) is the same as $(-r, \theta + \pi)$.

The inverse functions are called the arcsine, arccosine, and arctangent, respectively. There are arithmetic relations between these functions, which are known as trigonometric identities.

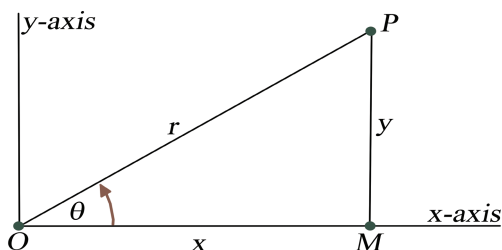


Figure 6: Cartesian and polar coordinates

The point labeled P in the figure resides in the plane. Therefore, it requires two dimensions to fix its location, either in Cartesian coordinates (x, y) or in polar coordinates (r, θ) .

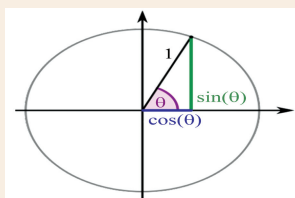
Given the Cartesian equation for a curve, the polar equation for the same curve can be obtained in terms of the radius r and the angle θ by substituting $r \cos \theta$ and $r \sin \theta$ for x and y , respectively. For example, the circle $x^2 + y^2 = a^2$ has the polar equation $(r \cos \theta)^2 + (r \sin \theta)^2 = a^2$, which reduces to $r = a$. (The positive value of r is sufficient, if θ takes all values from $-\pi$ to π or from 0 to 2π). Thus the polar equation of a circle simply expresses the fact that the curve is independent of θ and has constant radius. In a similar manner, the line $y = x \tan \phi$ has the polar equation $\sin \theta = \cos \theta \tan \phi$, which reduces to $\theta = \phi$. (The other solution, $\theta = \phi + \pi$, can be discarded if r is allowed to take negative values.)

Transformation of coordinates

A transformation of coordinates in a plane is a change from one coordinate system to another. Thus, a point in the plane will have two sets of coordinates giving its position with respect to the two coordinate systems used, and a transformation will express the relationship between the coordinate systems.

In a translation of Cartesian coordinate axes, a transformation is made between two sets of axes that are parallel to each other but have their origins at different positions. If a point P has coordinates (x, y) in one system, its coordinates in the second system are given by $(x - h, y - k)$ where (h, k) is the origin of the second system in terms of the first coordinate system. Thus, the transformation of P between the first system (x, y) and the second system (x', y') is given by the equations $x = x' + h$ and $y = y' + k$. The common use of translations of axes is to simplify the equations of curves. For example, the equation $2x^2 + y^2 - 12x - 2y + 17 = 0$ can be simplified with the translations $x' = x - 3$ and $y' = y - 1$ to an equation involving only squares of the variables and a constant term: $(x')^2 + (y')^2/2 = 1$. In other words, the curve represents an ellipse with its centre at the point $(3, 1)$ in the original coordinate system.

A rotation of coordinate axes is one in which a pair of axes giving the coordinates of a point (x, y) rotate through an angle ϕ to give a new pair of axes in which the point has coordinates (x', y') , as shown in the figure. The transformation equations for such a rotation are given by $x = x' \cos \phi - y' \sin \phi$ and $y = x' \sin \phi + y' \cos \phi$. The application of these formulas with $\phi = 45^\circ$ to the difference of squares, $x^2 - y^2 = a^2$, leads to the equation $x'y' = c$ (where c is a constant that depends on the value of a). This equation gives the form of the rectangular hyperbola when its asymptotes (the lines that a curve approaches without ever quite meeting) are used as the coordinate axes.

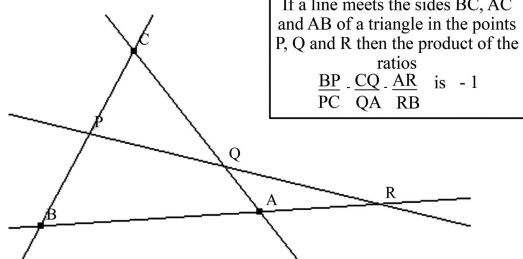


CORE CONCEPT: TRIGONOMETRIC FUNCTIONS

Trigonometric functions (also called circular functions, angle functions or goniometric functions) are functions of an angle. They relate the angles of a triangle to the lengths of its sides. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.

The first work on trigonometric functions related to chords of a circle. Given a circle of fixed radius, 60 units were often used in early calculations, then the problem was to find the length of the chord subtended by a given angle. For a circle of unit radius the length of the chord subtended by the angle x was $2\sin(x/2)$. The first known table of chords was produced by the Greek mathematician Hipparchus in about 140 BC. Although these tables have not survived, it is claimed that twelve books of tables of chords were written by Hipparchus. This makes Hipparchus the founder of trigonometry. The next Greek mathematician to produce a table of chords was Menelaus in about 100 AD. Menelaus worked in Rome producing six books of tables of chords which have been lost but his work on spherics has survived and is the earliest known work on spherical trigonometry. Menelaus proved a property of plane triangles and the corresponding spherical triangle property known the *regula sex quantitatium*.

Menelaus's Theorem



Ptolemy was the next author of a book of chords, showing the same Babylonian influence as Hipparchus, dividing the circle into 360° and the diameter into 120 parts. The suggestion here is that he was following earlier practice when the approximation 3 for π was used. Ptolemy, together with the earlier writers, used a form of the relation $\sin^2 x + \cos^2 x = 1$, although of course they did not actually use sines and cosines but chords.

Similarly, in terms of chords rather than sin and cos, Ptolemy knew the formulas
 $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$a/\sin A = b/\sin B = c/\sin C.$$

Ptolemy calculated chords by first inscribing regular polygons of 3, 4, 5, 6 and 10 sides in a circle. This allowed him to calculate the chord subtended by angles of 36° , 72° , 60° , 90° and 120° . He then found a method of finding the cord subtended by half the arc of a known chord and this, together with interpolation allowed him to calculate chords with a good degree of accuracy. Using these methods Ptolemy found that $\sin 30'$ ($30' = \text{half of } 1^\circ$) which is the chord of 1° was, as a number to base 60, 0 31' 25". Converted to decimals this is 0.0087268 which is correct to 6 decimal places, the answer to 7 decimal places being 0.0087265.

The first actual appearance of the sine of an angle appears in the work of the Hindus. Aryabhata, in about 500, gave tables of half chords which now really are sine tables and used *jya* for our $\sin$. This same table was reproduced in the work of Brahmagupta (in 628) and detailed method for constructing a table of sines for any angle were give by Bhaskara in 1150.

The Arabs worked with sines and cosines and by 980 Abu'l-Wafa knew that

$$\sin 2x = 2 \sin x \cos x$$

although it could have easily have been deduced from Ptolemy's formula $\sin(x + y) = \sin x \cos y + \cos x \sin y$ with $x = y$.

The Hindu word *jya* for the sine was adopted by the Arabs who called the sine *jiba*, a meaningless word with the same sound as *jya*. Now *jiba* became *jaib* in later Arab writings and this word does have a meaning, namely a 'fold'. When European authors translated the Arabic mathematical works into Latin they translated *jaib* into the word *sinus* meaning fold in Latin. In particular Fibonacci's use of the term *sinus rectus arcus* soon encouraged the universal use of sine.

Copernicus's book giving all the trigonometry relevant to astronomy was published in 1542 by Rheticus. Rheticus also produced substantial tables of sines and cosines which were published after his death. In 1533 Regiomontanus's work *De triangulis omnimodis* was published. This contained work on planar and spherical trigonometry originally done much earlier in about 1464. The book is particularly strong on the sine and its inverse.

The term sine certainly was not accepted straight away as the standard notation by all authors. In times when mathematical notation was in itself a new idea many used their own notation. Edmund Gunter was the first to use the abbreviation $\sin$ in 1624 in a drawing. The first use of $\sin$ in a book was in 1634 by the French mathematician Hérigone while Cavalieri used *Si* and Oughtred *S*.

It is perhaps surprising that the second most important trigonometrical function during the period we have discussed was the versed sine, a function now hardly used at all. The versine is related to the sine by the formula

$$\text{versin } x = 1 - \cos x.$$

It is just the sine turned (versed) through 90° .

The cosine follows a similar course of development in notation as the sine. Viète used the term *sinus residuae* for the cosine, Gunter (1620) suggested co-sinus. The notation Si.2 was used by Cavalieri, s co arc by Oughtred and S by Wallis.

Viète knew formulas for $\sin nx$ in terms of $\sin x$ and $\cos x$. He gave explicitly the formulas (due to Pitiscus)

$$\sin 3x = 3 \cos^2 x \sin x - \sin^3 x$$

$$\cos 3x = \cos^3 x - 3 \sin^2 x \cos x.$$

The tangent and cotangent came via a different route from the chord approach of the sine. These developed together and were not at first associated with angles. They became important for calculating heights from the length of the shadow that the object cast. The length of shadows was also of importance in the sundial. Thales used the lengths of shadows to calculate the heights of pyramids.

The first known tables of shadows were produced by the Arabs around 860 and used two measures translated into Latin as *umbra recta* and *umbra versa*. Viète used the terms *amsinus* and *prosinus*. The name tangent was first used by Thomas Fincke in 1583. The term cotangens was first used by Edmund Gunter in 1620.

Abbreviations for the tan and cot followed a similar development to those of the sin and cos. Cavalieri used Ta and Ta.2, Oughtred used t arc and t co arc while Wallis used T and t . The common abbreviation used today is tan by we write tan whereas the first occurrence of this abbreviation was used by Albert Girard in 1626, but tan was written over the angle cot was first used by Jonas Moore in 1674.

tan

A

The secant and cosecant were not used by the early astronomers or surveyors. These came into their own when navigators around the 15th Century started to prepare tables. Copernicus knew of the secant which he called the **hypotenusa**. Viète knew the results

$$\operatorname{cosec} x / \sec x = \cot x = 1 / \tan x$$

$$1 / \operatorname{cosec} x = \cos x / \cot x = \sin x.$$

Hypotenuse is the longest side of a right-angled triangle, the side opposite the right angle.

The abbreviations used by various authors were similar to the trigonometric functions already discussed. Cavalieri used Se and $Se.2$, Oughtred used se arc and sec co arc while Wallis used s and σ . Albert Girard used sec , written above the angle as he did for the $\tan$.

The term ‘trigonometry’ first appears as the title of a book *Trigonometria* by B Pitiscus, published in 1595. Pitiscus also discovered the formulas for $\sin 2x$, $\sin 3x$, $\cos 2x$, $\cos 3x$. The 18th Century saw trigonometric functions of a complex variable being studied. Johann Bernoulli found the relation between $\sin^{-1}z$ and $\log z$ in 1702 while Cotes, in a work published in 1722 after his death, showed that $ix = \log(\cos x + i \sin x)$.

De Moivre published his famous theorem

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx$$

in 1722 while Euler, in 1748, gave the formula (equivalent to that of Cotes)

$$\exp(ix) = \cos x + i \sin x.$$

The hyperbolic trigonometric functions were introduced by Lambert.

Relationship Between Degrees and Radians

A radian is defined as an angle θ subtended at the center of a circle for which the arc length is equal to the radius of that circle (Figure 7).

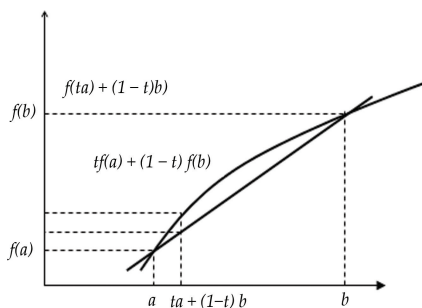


Figure 7: Definition of a radian.

The circumference of the circle is equal to $2\pi R$, where R is the radius of the circle. Consequently, $360^\circ = 2\pi$ radians. Thus,

$$1 \text{ radian} = 360^\circ / 2\pi \approx 57.296^\circ$$

$$1^\circ = (2\pi / 360) \text{ radians} \approx 0.01745 \text{ radians}$$

The Unit Circle

In mathematics, a unit circle is defined as a circle with a radius of 1. Often, especially in applications to trigonometry, the unit circle is centered at the origin $(0,0)$ in the coordinate plane. The equation of the unit circle in the coordinate plane is

$$x^2 + y^2 = 1.$$

As mentioned above, the unit circle is taken to be 360° , or 2π radians. We can divide the coordinate plane, and therefore, the unit circle, into 4 quadrants. The first quadrant is defined in terms of coordinates by $x > 0$, $y > 0$, or, in terms of angles, by $0^\circ < \theta < 90^\circ$, or $0 < \theta < \frac{\pi}{2}$.

Definitions of Trigonometric Functions for a Right Triangle

A right triangle is a triangle with a right angle (90°) (Figure 8).

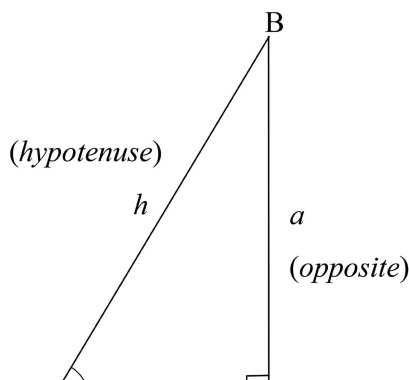


Figure.8: Right triangle.

For every angle θ in the triangle, there is the side of the triangle adjacent to it (from here on denoted as “adj”), the side opposite of it (from here on denoted as “opp”), and the hypotenuse (from here on denoted as “hyp”), which is the longest side of the triangle located opposite of the right angle. For angle θ , the trigonometric functions are defined as follows:

$$\text{sine of } \theta = \sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\text{cosine of } \theta = \cos \theta = \frac{\text{adj}}{\text{hyp}}$$

$$\text{tangent of } \theta = \tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}}$$

$$\text{cotangent of } \theta = \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta} = \frac{\text{adj}}{\text{opp}}$$

$$\text{secant of } \theta = \sec \theta = \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}}$$

$$\text{cosecant of } \theta = \csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}}$$

Definitions of Trigonometric Functions for a Unit Circle

In the unit circle, one can define the trigonometric functions cosine and sine as follows. If (x, y) is a point on the unit circle, and if the ray from the origin $(0, 0)$ to that

point (x,y) makes an angle θ with the positive x -axis, (such that the counterclockwise direction is considered positive), then,

$$\cos\theta = x/1 = x$$

$$\sin\theta = y/1 = y$$

Then, each point (x,y) on the unit circle can be written as $(\cos\theta, \sin\theta)$. Combined with the equation $x^2 + y^2 = 1$, the definitions above give the relationship $\sin^2\theta + \cos^2\theta = 1$. In addition, other trigonometric functions can be defined in terms of x and y :

$$\tan\theta = \sin\theta/\cos\theta = y/x$$

$$\cot\theta = \cos\theta/\sin\theta = x/y$$

$$\sec\theta = 1/\cos\theta = 1/x$$

$$\csc\theta = 1/\sin\theta = 1/y$$

Figure 9 below shows a unit circle in the coordinate plane, together with some useful values of angle θ , and the points $(x,y)=(\cos\theta, \sin\theta)$, that are most commonly used.

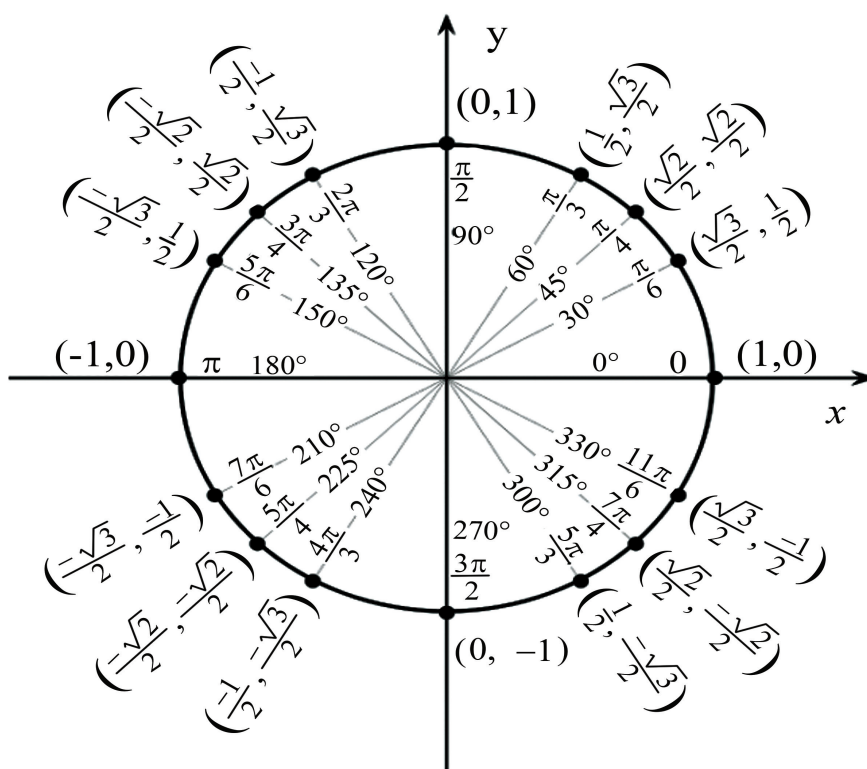


Figure 9. Most commonly used angles and points of the unit circle.

Note: For θ in quadrant I, $\sin\theta > 0$, $\cos\theta > 0$; for θ in quadrant II, $\sin\theta > 0$, $\cos\theta < 0$

Exact Values for Trigonometric Functions of Most Commonly Used Angles

θ in degrees	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	undefined
180	π	0	-1	0
270	$\frac{3\pi}{2}$	-1	0	undefined
360	2π	0	1	0

Trigonometric Functions of Any Angle θ in Terms of Angle θ in Quadrant I

θ'	$\sin\theta'$	$\cos\theta'$	$\tan\theta'$	θ'	$\sin\theta'$	$\cos\theta'$	$\tan\theta'$
$90^\circ + \theta$ $\pi/2 + \theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$	$90^\circ - \theta$ $\pi/2 - \theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$
$180^\circ + \theta$ $\pi + \theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$	$180^\circ - \theta$ $\pi - \theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$
$270^\circ + \theta$ $3\pi/2 + \theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$	$270^\circ - \theta$ $3\pi/2 - \theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$
$k(360^\circ) + \theta$ $k(2\pi) + \theta$ $k = \text{integer}$	$\sin\theta$	$\cos\theta$	$\tan\theta$	$k(360^\circ) - \theta$ $k(2\pi) - \theta$ $k = \text{integer}$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$

Trigonometric Functions of Negative Angles

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

Some Useful Relationships Among Trigonometric Functions

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

Double Angle Formulas

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2 \sin^2\theta = 2 \cos^2\theta - 1$$

$$\tan 2\theta = 2 \frac{\tan\theta}{1 - \tan^2\theta}$$

Half Angle Formulas

Note: in the formulas in this section, the “+” sign is used in the quadrants where the respective trigonometric function is positive for angle $\theta/2$, and the “-” sign is used in the quadrants where the respective trigonometric function is negative for angle $\theta/2$.

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

Angle Addition Formulas

Note: in this and the following section, letters A and B are used to denote the angles of interest, instead of the letter θ .

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$$

Sum, Difference and Product of Trigonometric Functions

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sin A \sin B = \frac{1}{2} [\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A-B) + \cos(A+B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A-B) + \sin(A+B)]$$

Graphs of Trigonometric Functions

In each graph in Figure 10, the horizontal axis (x) is measured in radians

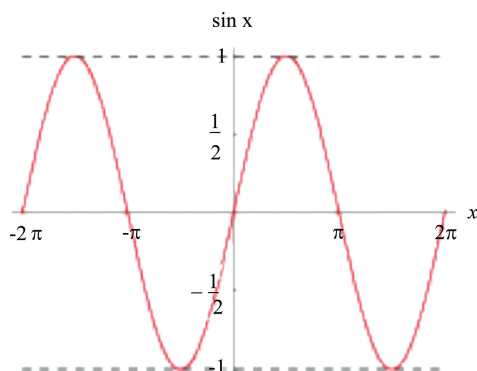


Figure 10a: Graph of $\sin(x)$.

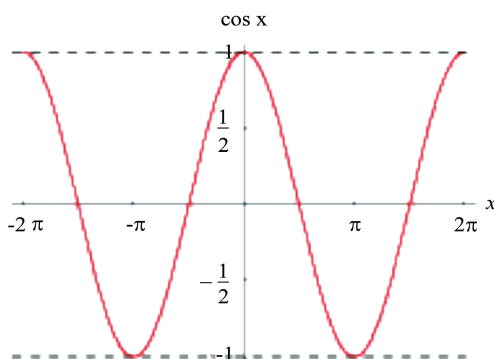


Figure 10b: Graph of $\cos(x)$.

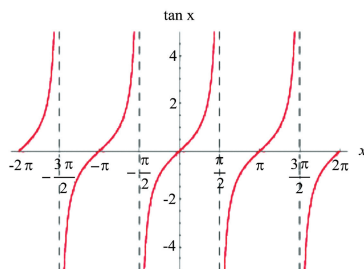


Figure 10c: Graph of $\tan(x)$.

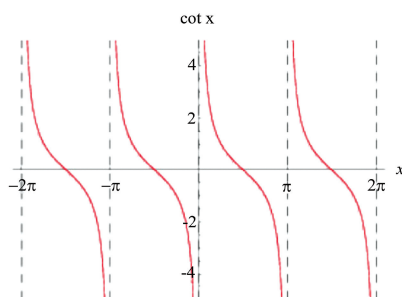


Figure 10d: Graph of $\cot(x)$.

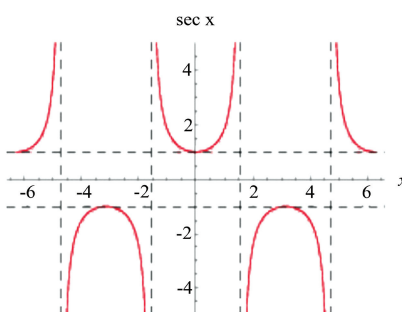


Figure 10e: Graph of $\sec(x)$.

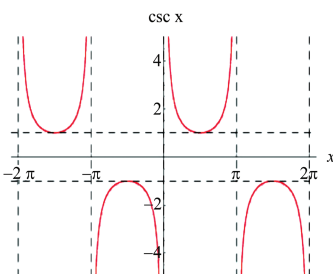
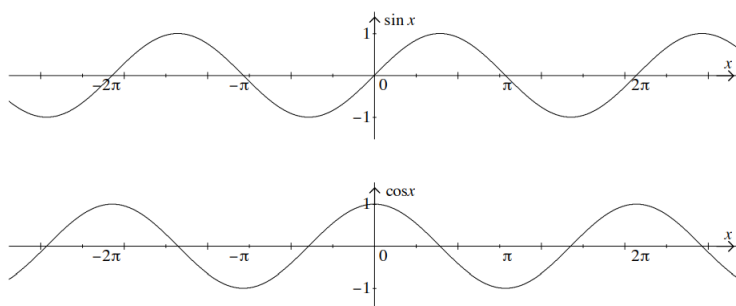


Figure 10f: Graph of $\csc(x)$.

The graphs $y = \sin x$ and $y = \cos x$ to sketch the graphs of more complex trigonometric functions.



Let's look first at some important features of these two graphs.

The shape of each graph is repeated after every interval of length 2π .

This makes sense when we think of the way we have defined $\sin$ and $\cos$ using the unit circle.

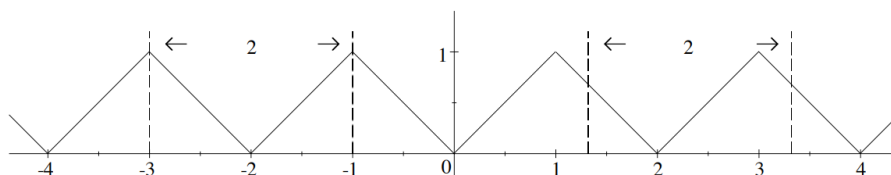
We say that these functions are periodic with period 2π .

The $\sin$ and $\cos$ functions are the most famous examples of a class of functions called periodic functions.

Functions with the property that $f(x) = f(x + a)$ for all x are called periodic functions. Such a function is said to have period a .

This means that the function repeats itself after every interval of length a .

Note that you can have periodic functions that are not trigonometric functions. For example, the function below is periodic with period 2.



The values of the functions $y = \sin x$ and $y = \cos x$ oscillate between -1 and 1 . We say that $y = \sin x$ and $y = \cos x$ have amplitude 1. A general definition for the amplitude of any periodic function is:

The amplitude of a periodic function is half the distance between its minimum and maximum values.

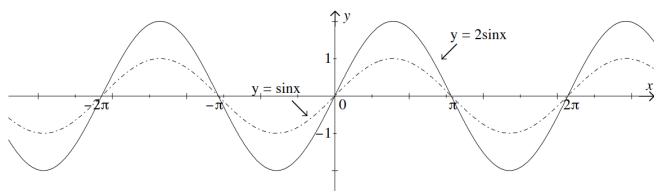
Also, the functions $y = \sin x$ and $y = \cos x$ oscillates about the x -axis. We refer to the x -axis as the mean level of these functions, or say that they have a mean level of 0

We notice that the graphs of $\sin x$ and $\cos x$ have the same shape. The graph of $\sin x$ looks like the graph of $\cos x$ shifted to the right by $\frac{\pi}{2}$ units. We say that the phase difference between the two functions is $\frac{\pi}{2}$.

Other trigonometric functions can be obtained by modifying the graphs of $\sin x$ and $\cos x$.

Changing the amplitude

Consider the graph of the function $y = 2 \sin x$.



The graph of $y = 2 \sin x$ has the same period as $y = \sin x$ but has been stretched in the y direction by a factor of 2. That is, for every value of x the y value for $y = 2 \sin x$ is twice the y value for $y = \sin x$.

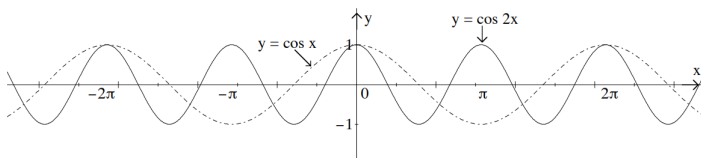
So, the amplitude of the function $y = 2 \sin x$ is 2. Its period is 2π .

In general we can say that the amplitude of the function $y = a \sin x$ is a , since in this case $y = a \sin x$ oscillates between $-a$ and a .

Changing the period

Let's consider the graph of $y = \cos 2x$.

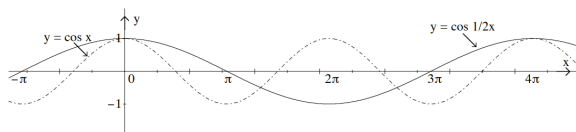
To sketch the graph of $y = \cos 2x$, first think about some specific points. We will look at the points where the function $y = \cos x$ equals 0 or ± 1 .



$\cos x = 1$	when $x = 0$,	so $\cos 2x = 1$	when $2x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos 2x = 0$	when $2x = \frac{\pi}{2}$,	ie $x = \frac{\pi}{4}$.
$\cos x = -1$	when $x = \pi$,	so $\cos 2x = -1$	when $2x = \pi$,	ie $x = \frac{\pi}{2}$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos 2x = 0$	when $2x = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{4}$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos 2x = 1$	when $2x = 2\pi$,	ie $x = \pi$.

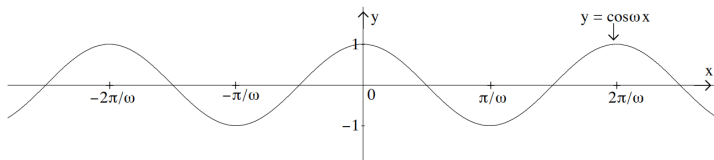
As we see from the graph, the function $y = \cos \frac{1}{2}x$ has a period of π . The function still oscillates between the values -1 and 1 , so its amplitude is 1.

Now, let's consider the function $y = \cos 1/2x$. Again we will sketch the graph by looking at the points where $y = \cos x$ equals 0 or ± 1 .



$\cos x = 1$	when $x = 0$,	so $\cos \frac{1}{2}x = 1$	when $\frac{1}{2}x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos \frac{1}{2}x = 0$	when $\frac{1}{2}x = \frac{\pi}{2}$,	ie $x = \pi$.
$\cos x = -1$	when $x = \pi$,	so $\cos \frac{1}{2}x = -1$	when $\frac{1}{2}x = \pi$,	ie $x = 2\pi$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos \frac{1}{2}x = 0$	when $\frac{1}{2}x = \frac{3\pi}{2}$,	ie $x = 3\pi$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos \frac{1}{2}x = 1$	when $\frac{1}{2}x = 2\pi$,	ie $x = 4\pi$.

In this case our modified function $y = \cos \frac{1}{2}x$ has period 4π . Its amplitude is 1. What happens if we take the function $y = \cos \omega x$ where $\omega > 0$?



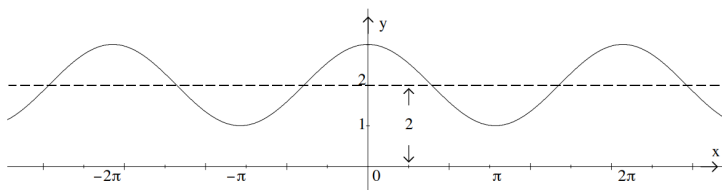
$\cos x = 1$	when $x = 0$,	so $\cos \omega x = 1$	when $\omega x = 0$,	ie $x = 0$.
$\cos x = 0$	when $x = \frac{\pi}{2}$,	so $\cos \omega x = 0$	when $\omega x = \frac{\pi}{2}$,	ie $x = \frac{\pi}{2\omega}$.
$\cos x = -1$	when $x = \pi$,	so $\cos \omega x = -1$	when $\omega x = \pi$,	ie $x = \frac{\pi}{\omega}$.
$\cos x = 0$	when $x = \frac{3\pi}{2}$,	so $\cos \omega x = 0$	when $\omega x = \frac{3\pi}{2}$,	ie $x = \frac{3\pi}{2\omega}$.
$\cos x = 1$	when $x = 2\pi$,	so $\cos \omega x = 1$	when $\omega x = 2\pi$,	ie $x = \frac{2\pi}{\omega}$.

In general, if we take the function $y = \cos \omega x$ where $\omega > 0$, the period of the function is $\frac{2\pi}{\omega}$.

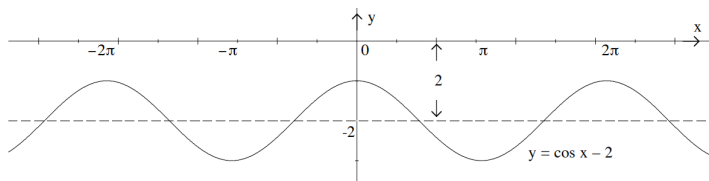
Changing the mean level

We saw above that the functions $y = \sin x$ and $y = \cos x$ both oscillate about the x -axis which is sometimes referred to as the mean level for $y = \sin x$ and $y = \cos x$.

We can change the mean level of the function by adding or subtracting a constant. For example, adding the constant 2 to $y = \cos x$ gives us $y = \cos x + 2$ and has the effect of shifting the whole graph up by 2 units. So, the mean level of $y = \cos x + 2$ is 2.



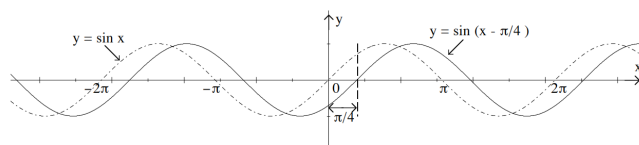
Similarly, we can shift the graph of $y = \cos x$ down by two units. In this case, we have $y = \cos x - 2$, and this function has mean level -2 .



In general, if $d > 0$, the function $y = \cos x + d$ looks like the function $y = \cos x$ shifted up by d units. If $d < 0$, then the function $y = \cos x - d$ looks like the function $y = \cos x$ shifted down by d units. If $d < 0$, say $d = -2$, the function $y = \cos x + d = \cos x + (-2)$ can be written as $y = \cos x - 2$ so again looks like the function $y = \cos x$ shifted down by 2 units.

Changing the phase

Consider the function $y = \sin(x - \frac{\pi}{4})$.



To sketch the graph of $y = \sin(x - \frac{\pi}{4})$ we again use the points at which $y = \sin x$ is 0 or ± 1 .

$$\sin x = 0 \quad \text{when } x = 0, \quad \text{so } \sin(x - \frac{\pi}{4}) = 0 \quad \text{when } x - \frac{\pi}{4} = 0, \quad \text{ie } x = 0 + \frac{\pi}{4} = \frac{\pi}{4}.$$

$$\sin x = 1 \quad \text{when } x = \frac{\pi}{2}, \quad \text{so } \sin(x - \frac{\pi}{4}) = 1 \quad \text{when } x - \frac{\pi}{4} = \frac{\pi}{2}, \quad \text{ie } x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}.$$

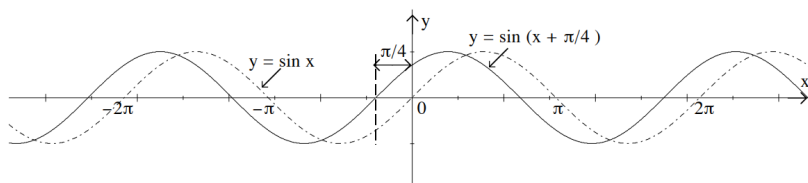
$$\sin x = 0 \quad \text{when } x = \pi, \quad \text{so } \sin(x - \frac{\pi}{4}) = 0 \quad \text{when } x - \frac{\pi}{4} = \pi, \quad \text{ie } x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}.$$

$$\sin x = -1 \quad \text{when } x = \frac{3\pi}{2}, \quad \text{so } \sin(x - \frac{\pi}{4}) = -1 \quad \text{when } x - \frac{\pi}{4} = \frac{3\pi}{2}, \quad \text{ie } x = \frac{3\pi}{2} + \frac{\pi}{4} = \frac{7\pi}{4}.$$

$$\sin x = 0 \quad \text{when } x = 2\pi, \quad \text{so } \sin(x - \frac{\pi}{4}) = 0 \quad \text{when } x - \frac{\pi}{4} = 2\pi, \quad \text{ie } x = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}.$$

The graph of $y = \sin(x - \frac{\pi}{4})$ looks like the graph of $y = \sin x$ shifted $\frac{\pi}{4}$ units to the right. We say that there has been a phase shift to the right by $\frac{\pi}{4}$.

Now consider the function $y = \sin(x + \frac{\pi}{4})$.



$$\sin x = 0 \quad \text{when } x = 0, \quad \text{so } \sin(x + \frac{\pi}{4}) = 0 \quad \text{when } x + \frac{\pi}{4} = 0, \quad \text{ie } x = 0 - \frac{\pi}{4} = -\frac{\pi}{4}.$$

$$\sin x = 1 \quad \text{when } x = \frac{\pi}{2}, \quad \text{so } \sin(x + \frac{\pi}{4}) = 1 \quad \text{when } x + \frac{\pi}{4} = \frac{\pi}{2}, \quad \text{ie } x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

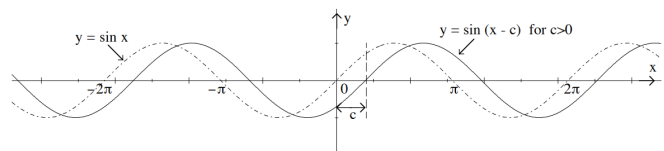
$$\sin x = 0 \quad \text{when } x = \pi, \quad \text{so } \sin(x + \frac{\pi}{4}) = 0 \quad \text{when } x + \frac{\pi}{4} = \pi, \quad \text{ie } x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

$$\sin x = -1 \quad \text{when } x = \frac{3\pi}{2}, \quad \text{so } \sin(x + \frac{\pi}{4}) = -1 \quad \text{when } x + \frac{\pi}{4} = \frac{3\pi}{2}, \quad \text{ie } x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}.$$

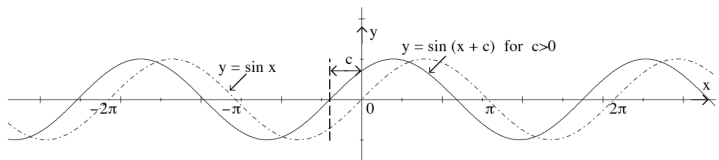
$$\sin x = 0 \quad \text{when } x = 2\pi, \quad \text{so } \sin(x + \frac{\pi}{4}) = 0 \quad \text{when } x + \frac{\pi}{4} = 2\pi, \quad \text{ie } x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}.$$

The graph of $y = \sin(x + \frac{\pi}{4})$ looks like the graph of $y = \sin x$ shifted $\frac{\pi}{4}$ units to the left.

More generally, the graph of $y = \sin(x - c)$ where $c > 0$ can be drawn by shifting the graph of $y = \sin x$ to the right by c units.



Similarly, the graph of $y = \sin(x + c)$ where $c > 0$ can be drawn by shifting the graph of $y = \sin x$ to the left by c units.



What happens to $y = \sin(x - c)$ if $c < 0$? Consider, for example, what happens when $c = \frac{\pi}{4}$. In this case we can write $x - \frac{\pi}{4} = x + \frac{\pi}{4}$. So, $y = \sin(x - \frac{\pi}{4}) = \sin(x + \frac{\pi}{4})$ and we have a shift to the left by $\pi/4$ units as before

Inverse Trigonometric Functions

$x = \sin(y)$	$y = \sin^{-1}(x) = \arcsin(x)$
$x = \cos(y)$	$y = \cos^{-1}(x) = \arccos(x)$
$x = \tan(y)$	$y = \tan^{-1}(x) = \arctan(x)$
$x = \cot(y)$	$y = \cot^{-1}(x) = \text{arccot}(x)$
$x = \sec(y)$	$y = \sec^{-1}(x) = \text{arcsec}(x)$
$x = \csc(y)$	$y = \csc^{-1}(x) = \text{arccsc}(x)$

Principal Values for Inverse Trigonometric Functions

Principal Values for $x \geq 0$	Principal Values for $x < 0$
$0 \leq \sin^{-1}(x) \leq \pi/2$	$-\pi/2 \leq \sin^{-1}(x) < 0$
$0 \leq \cos^{-1}(x) \leq \pi/2$	$\pi/2 < \cos^{-1}(x) \leq \pi$
$0 \leq \tan^{-1}(x) < \pi/2$	$-\pi/2 < \tan^{-1}(x) < 0$
$0 < \cot^{-1}(x) \leq \pi/2$	$\pi/2 < \cot^{-1}(x) < \pi$
$0 \leq \sec^{-1}(x) < \pi/2$	$\pi/2 < \sec^{-1}(x) \leq \pi$
$0 < \csc^{-1}(x) \leq \pi/2$	$-\pi/2 \leq \csc^{-1}(x) < 0$

Relations Between Inverse Trigonometric Functions

Note: In all cases, it is assumed that principal values are used.

$\sin^{-1}x + \cos^{-1}x = \pi/2$

$\tan^{-1}x + \cot^{-1}x = \pi/2$

$$\sec^{-1}x + \csc^{-1}x = \pi/2$$

$$\csc^{-1}x = \sin^{-1}(1/x)$$

$$\sec^{-1}x = \cos^{-1}(1/x)$$

$$\cot^{-1}x = \tan^{-1}(1/x)$$

$$\sin^{-1}(-x) = -\sin^{-1}x$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}x$$

$$\tan^{-1}(-x) = -\tan^{-1}x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1}x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1}x$$

$$\csc^{-1}(-x) = -\csc^{-1}x$$

Graphs of Inverse Trigonometric Functions

In each graph in Figure 11, the vertical axis (y) is measured in radians. Only portions of curves corresponding to principal values are shown.

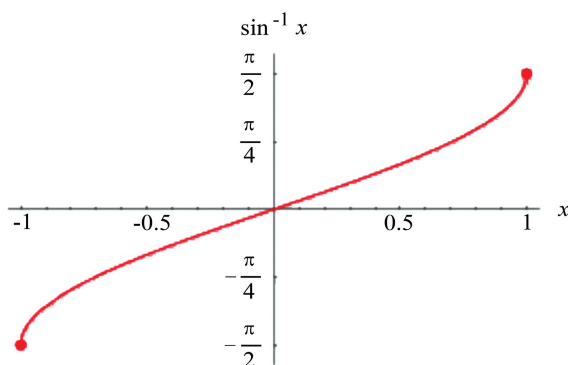


Figure 11a. Graph of $\sin^{-1}(x)$.

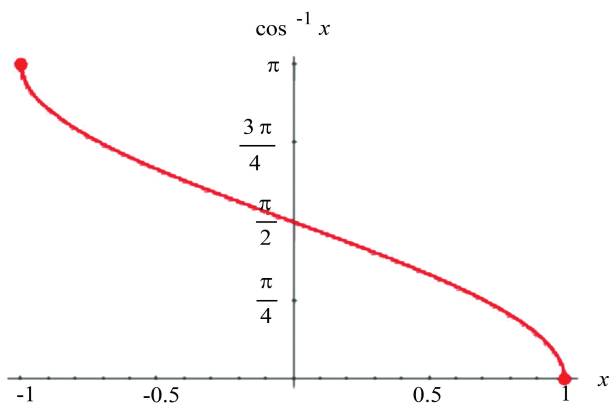


Figure 11b: Graph of $\cos^{-1}(x)$.

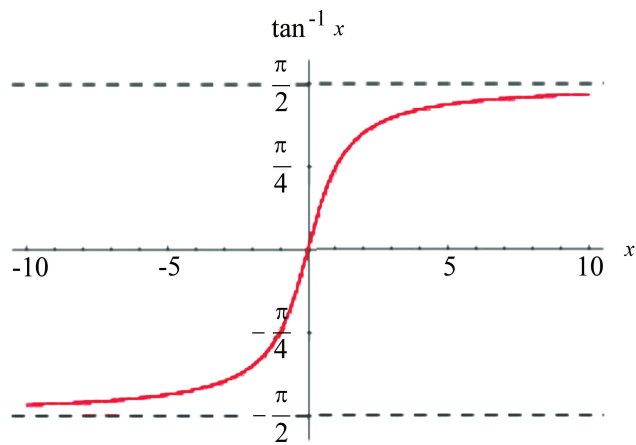


Figure 11c: Graph of $\tan^{-1}(x)$.

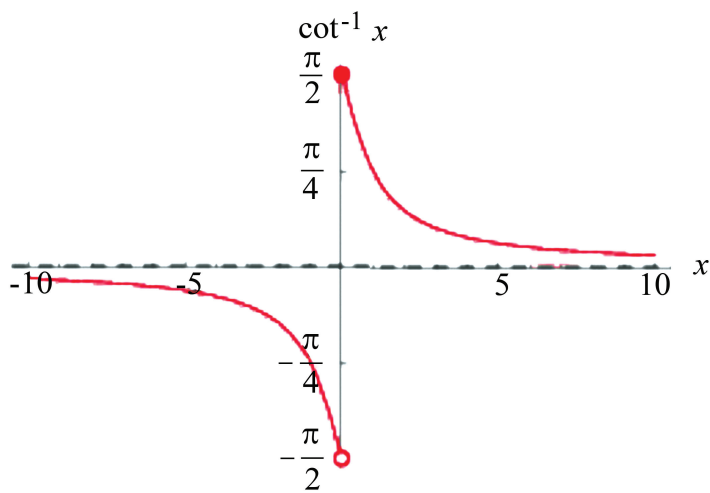


Figure 11d: Graph of $\cot^{-1}(x)$.

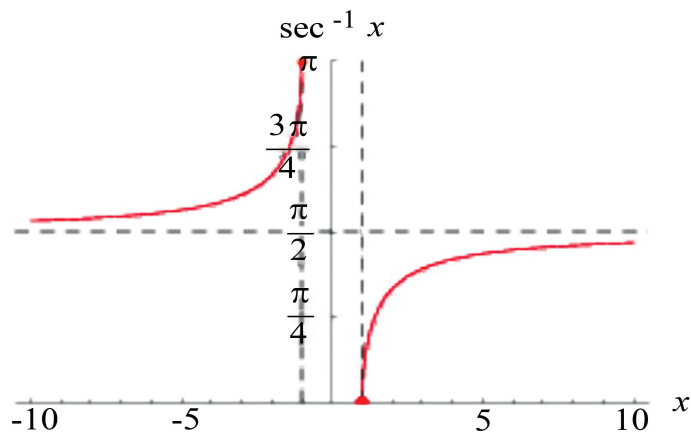


Figure 11e: Graph of $\sec^{-1}(x)$.

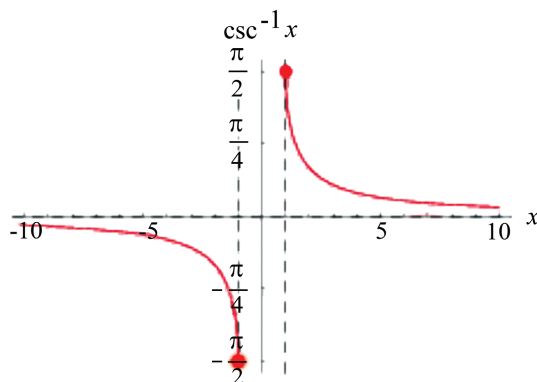


Figure 11f: Graph of $\csc^{-1}(x)$.

Using Trigonometric Functions

Resolving Vectors into Components:

The geometric way of adding vectors is not recommended whenever great accuracy is required or in three-dimensional problems. In such cases, it is better to make use of the projections of vectors along coordinate axis, also known as components of the vector. Any vector can be completely described by its components. Consider an arbitrary vector A (from now on, the bold-faced letters are used to signify vectors, whereas the regular-font letter A signifies the length of the vector A) lying in the xy -plane and making an arbitrary angle θ with the positive x -axis, as shown in Figure 12:

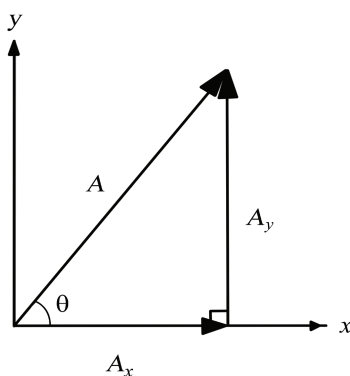


Figure 12: Arbitrary vector A in the xy -plane.

This vector A can be expressed as the sum of two other vectors, A_x and A_y . From Figure 12, it can be seen that the three vectors form a right triangle and that $A = A_x + A_y$. It is conventional to refer to the “components of vector A ,” written A_x and A_y (without the boldface notation). The component A_x represents the projection of A along the x -axis, and the component A_y represents the projection of A along the y -axis. These components can be positive or negative.

From Figure 12 and the definition of sine and cosine for a right triangle, we see that $\cos\theta = A_x/A$ and $\sin\theta = A_y/A$. Hence, the components of A are

$$A_x = A \cos \theta$$

$$A_y = A \sin \theta$$

These components form the two sides of a right triangle with hypotenuse of length A . Thus, it follows that the magnitude and direction of A are related to its components through the expressions

$$A = \sqrt{A_x^2 + A_y^2}$$

$$\theta = \tan^{-1} \left(\frac{A_y}{A_x} \right)$$

Note that the signs of the components A_x and A_y depend on the angle θ . For example, if $\theta = 120^\circ$, then A_x is negative and A_y is positive.

When solving problems, one can specify a vector A either with its components A_x and A_y or with its magnitude and direction A and θ . Furthermore, one can express a vector A as

$$A = A_x \hat{x} + A_y \hat{y} = A (\cos \theta \hat{x} + \sin \theta \hat{y})$$

where $\hat{x}$ and $\hat{y}$ are unit vectors (length of one) in the direction of x - and y -axis, respectively.

Writing a Phase Shift of a Wave

Waves can have many different shapes. One of the simplest to deal with and also one that is of a particular interest when it comes to simple harmonic motion is a sinusoidal wave. It is called “sinusoidal” because the shape that the wave takes either in space or in time is that of a sine curve (Figure 13a, b):

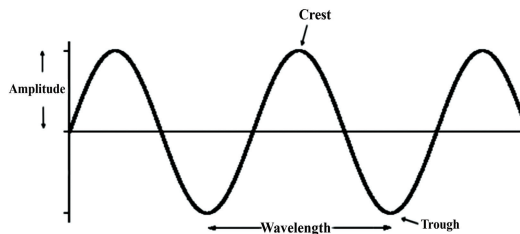


Figure 13a: A one-dimensional sinusoidal wave in space.

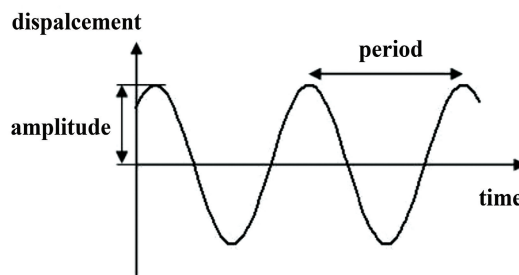


Figure 13b: A one-dimensional sinusoidal wave in time.

A function describing such a wave can be written as

$$y = A \sin\left(\frac{2\pi}{\lambda}x \pm \frac{2\pi}{T}t + \varphi\right)$$

where A is the amplitude of the wave, λ is the wavelength of the wave, T is the period of the wave, and φ is the phase shift of the wave. The $\pm$ sign in front of the second term in the parentheses depends on whether the wave is moving to the right (-) or to the left (+). The entire expression in the parentheses can also be written as $\Phi(x, t)$ and is called the total phase of the wave, and is a function of position x and time t , in contrast to the phase shift φ , which is a constant. Then, the entire expression becomes $y = A \sin \Phi(x, t)$.

Because the sine function is periodic with period 2π , $\sin(\Phi + 2\pi n) = \sin(\Phi)$ for any integer n . φ is referred to as a phase-shift, because it represents a “shift” from zero phase. But a change in φ is also referred to as a phase-shift. Figure 14 shows two curves: the red one with zero phase, and the blue one with a non-zero phase.

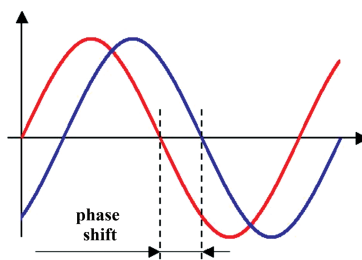


Figure 14: Phase shift.

Two oscillators that have the same frequency and same wavelength, will have a phase difference, if their phase shift φ is different. If that is the case, the oscillators are said to be out of phase with each other. The amount by which such oscillators are out of phase with each other can be expressed in degrees from 0° to 360° , or in radians from 0 to 2π (Figure 15a, b).

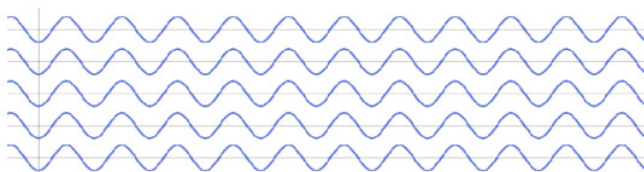


Figure 15a: Waves that are in phase.

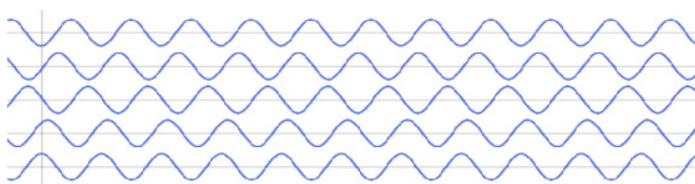


Figure 15b: Waves that are out of phase.

Derivatives of Trigonometric and Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -\frac{\pi}{2} < \sin^{-1} x < \frac{\pi}{2}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad 0 < \cos^{-1} x < \pi$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \quad -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}, \quad 0 < \cot^{-1} x < \pi$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{\pm 1}{x\sqrt{x^2-1}}, \quad \begin{array}{l} \text{+ if } 0 < \sec^{-1} x < \frac{\pi}{2} \\ \text{- if } \frac{\pi}{2} < \sec^{-1} x < \pi \end{array}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{\mp 1}{x\sqrt{x^2-1}}, \quad \begin{array}{l} \text{- if } 0 < \csc^{-1} x < \frac{\pi}{2} \\ \text{+ if } \frac{\pi}{2} < \csc^{-1} x < \pi \end{array}$$

Trigonometric Ratios and Identities

Trigonometric Ratios

Trigonometric functions are commonly defined as ratios of two sides of a right-angle triangle with one acute angle θ in standard position. Let the side opposite to that angle be called as opposite side and the other side be called as adjacent side.

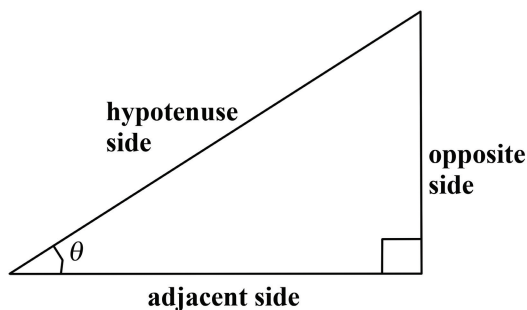


Figure 16

Thus,

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse side}}$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse side}}$$

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sin \theta}{\cos \theta}$$

$$\csc \theta = \frac{\text{hypotenuse side}}{\text{opposite side}} = \frac{1}{\sin \theta}$$

$$\sec \theta = \frac{\text{hypotenuse side}}{\text{adjacent side}} = \frac{1}{\cos \theta}$$

$$\cot \theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

Example

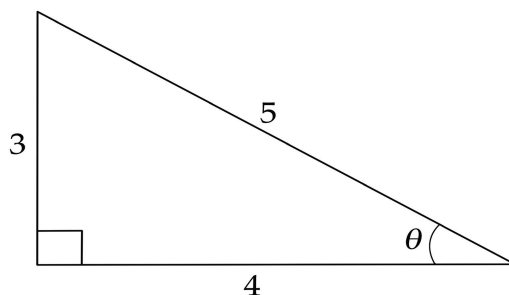
Given that $\cos \theta = \frac{4}{5}$, find the values of $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$ and $\cot \theta$.

Solution

A right-angle triangle below is obtained by using Theorem Pythagoras, $a^2 + b^2 = c^2$.

Hence,

$$\sin \theta = \frac{3}{5}, \quad \tan \theta = \frac{3}{4}, \quad \csc \theta = \frac{5}{3}, \quad \text{ü} \quad \theta = \frac{5}{4}, \quad \cot \theta = \frac{4}{3}$$



Based on a right-angle triangle, if one acute angle is θ , then the other angle is $90^\circ - \theta$ as shown in Figure 17.

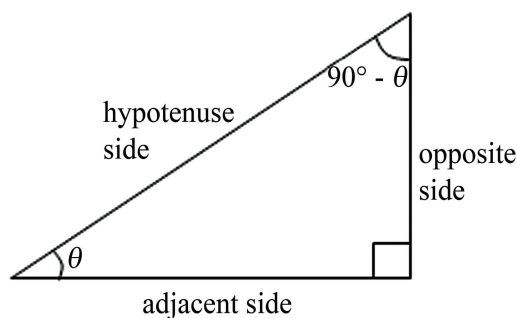


Figure 17

Thus,

$$\sin(90^\circ - \theta) = \frac{\text{adjacent side}}{\text{hypotenuse side}} = \cos \theta$$

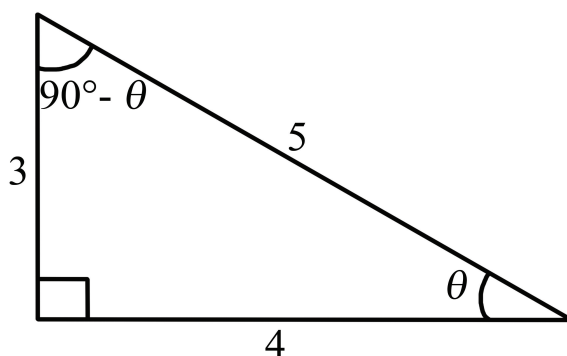
$$\cos(90^\circ - \theta) = \frac{\text{opposite side}}{\text{hypotenuse side}} = \sin \theta$$

$$\tan(90^\circ - \theta) = \frac{\text{adjacent side}}{\text{opposite side}} = \cot \theta$$

Example

Given that $\sin \theta = \frac{3}{5}$. Show that

- $\sin(90^\circ - \theta) = \cos \theta$
- $\cos(90^\circ - \theta) = \sin \theta$
- $\tan(90^\circ - \theta) = \cot \theta$



Solution:

Hence,

- $\sin(90^\circ - \theta) = \frac{4}{5} = \cos \theta$
- $\cos(90^\circ - \theta) = \frac{3}{5} = \sin \theta$

The value of the trigonometric ratio for any angle can be obtained from tables or calculators. For the acute angle 0° , 30° , 45° , 60° and 90° , the value of the trigonometric ratios of these angles can be described using an equilateral triangle and a right-angled isosceles, as shown in the table below.

Table 1

θ	0°	30°	45°	60°	90°
rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2} = 0.5$	$\frac{1}{\sqrt{2}} = 0.7071$	$\frac{\sqrt{3}}{2} = 0.866$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2} = 0.866$	$\frac{1}{\sqrt{2}} = 0.7071$	$\frac{1}{2} = 0.5$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}} = 0.5774$	1	$\sqrt{3} = 1.732$	∞

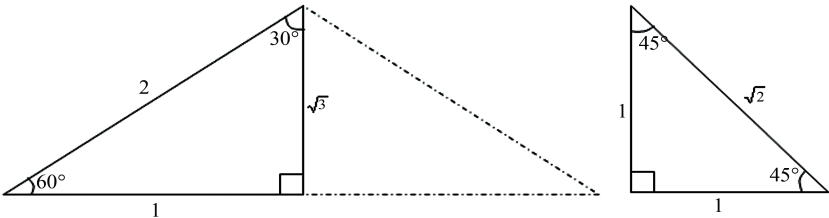


Figure 18

Figure 19 as below can be used to identify the sign of a trigonometric function of any angle.

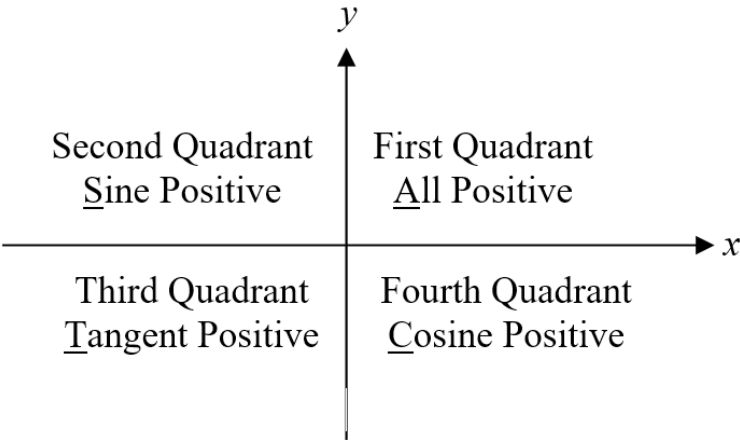


Figure 19

Example

Determine the basic angles of the following:

- 30°
- -220°
- 375°

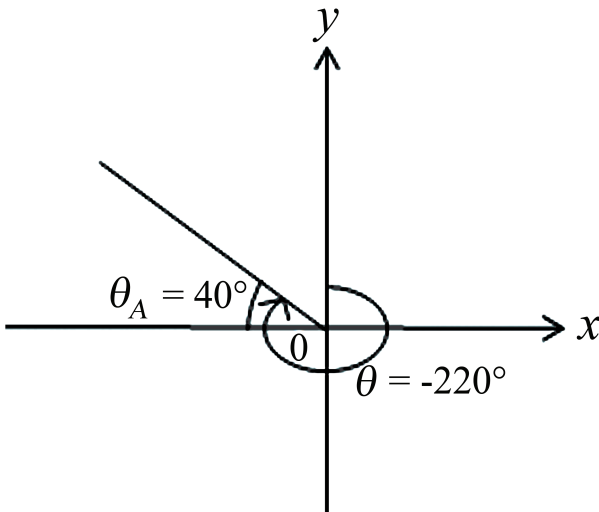
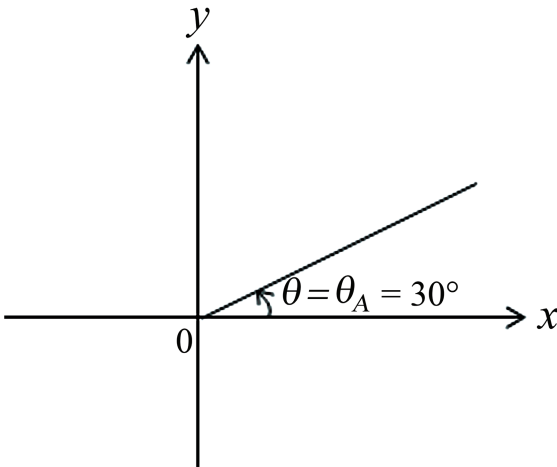
- -125°

Solution

- $\theta = 30^\circ$ is in the First Quadrant
- $\theta = -220^\circ$ is in the Second Quadrant.

Then,

$$\text{Then, } \theta_A = 220^\circ - 180^\circ = 40^\circ$$

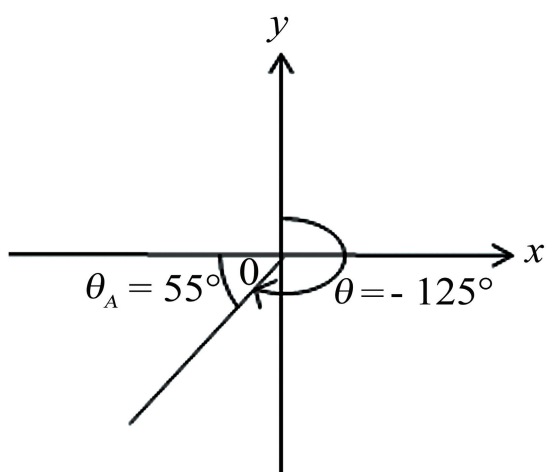
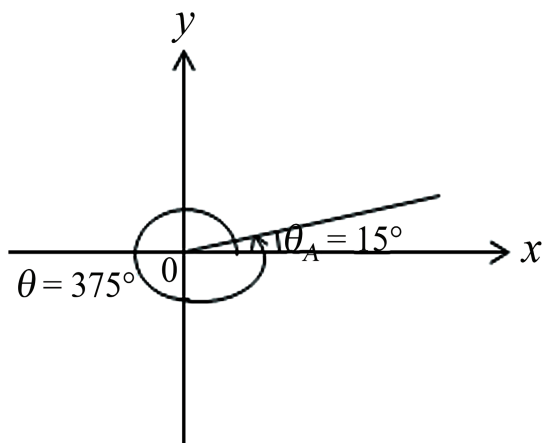


- $\theta = 375^\circ$ is in the First Quadrant.
- $\theta = -125^\circ$ is in the Third Quadrant.

$$\text{Then, } \theta_A = 375^\circ - 360^\circ = 15^\circ$$

$$\text{Then, } \theta_A = 180^\circ - 125^\circ = 55^\circ$$

Angle is the figure formed by two rays, called the sides of the angle, sharing a common endpoint, called the vertex of the angle.



CONCEPT OF LAW OF SINES

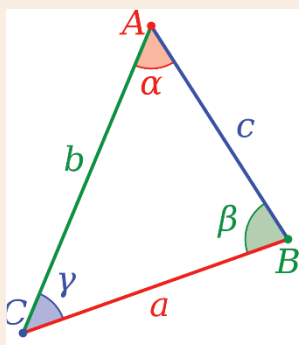
In trigonometry, the law of sines, sine law, sine formula, or sine rule is an equation relating the lengths of the sides of any triangle to the sines of its angles. According to the law,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R,$$

where a , b , and c are the lengths of the sides of a triangle, and α , β , and γ are the opposite angles, while R is the radius of the triangle's circumcircle. When the last part of the equation is not used, the law is sometimes stated using the reciprocals;

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

The law of sines can be used to compute the remaining sides of a triangle when two angles and a side are known—a technique known as triangulation. It can also be used when two sides and one of the non-enclosed angles are known. In some such cases, the triangle is not uniquely determined by this data (called the ambiguous case) and the technique gives two possible values for the enclosed angle.



The law of sines is one of two trigonometric equations commonly applied to find lengths and angles in scalene triangles, with the other being the law of cosines.

The law of sines can be generalized to higher dimensions on surfaces with constant curvature.

SUMMARY

- Trigonometric functions are also known as circular functions can be simply defined as the functions of an angle of a triangle.
- The trigonometric functions most widely used in modern mathematics are the sine, the cosine, and the tangent.
- There are a number of trigonometric formulas and identities that denotes the relation between the functions and help to find the angles of the triangle.
- Trigonometry developed from a need to compute angles and distances in such fields as astronomy, mapmaking, surveying, and artillery range finding.
- Various trigonometric identities show that the values of the functions for all angles can readily be found from the values for angles from 0° to 45° .
- The law of sines is expressed as an equality involving three sine functions while the law of cosines is an identification of the cosine with an algebraic expression formed from the lengths of sides opposite the corresponding angles.
- Spherical trigonometry involves the study of spherical triangles, which are formed by the intersection of three great circle arcs on the surface of a sphere.
- The angles of a spherical triangle are defined by the angle of intersection of the corresponding tangent lines to each vertex.
- Trigonometric functions of a real variable x are defined by means of the trigonometric functions of an angle.
- A transformation of coordinates in a plane is a change from one coordinate system to another.
- The geometric way of adding vectors is not recommended whenever great accuracy is required or in three-dimensional problems.

MULTIPLE CHOICE QUESTIONS

1. If $\sin \theta$ and $\cos \theta$ are the roots of $ax^2 - bx + c = 0$, then the relation between a , b and c will be
 - a. $a^2 - b^2 + 2ac = 0$
 - b. $a^2 + b^2 + 2ac = 0$
 - c. $a^2 + c^2 + 2ab = 0$
 - d. $a^2 - b^2 - 2ac = 0$
2. If $\tan A = 1/2$ and $\tan B = 1/3$, then the value of $A + B$ is
 - a. $\pi/6$
 - b. π
 - c. 0
 - d. $\pi/4$
3. The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 179^\circ$ is
 - a. $1/\sqrt{2}$
 - b. 0
 - c. 1
 - d. -1
4. The value of $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$ is equal to
 - a. 1
 - b. 0
 - c. $1/2$
 - d. 2
5. The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$ is
 - a. 0
 - b. $1/2$
 - c. 1
 - d. Not defined

Answer

1. (a) 2. (d) 3. (b) 4. (b) 5. (c)

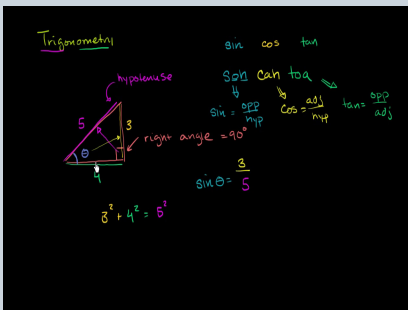
REVIEW QUESTIONS

1. A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find the angle in degrees which is subtended at the centre of hoop.
2. If $A = \cos 2\theta + \sin 4\theta$ for all values of θ , then prove that $\frac{3}{4} \leq A \leq 1$.
3. Find the value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$
4. Find the value of $\tan 9^\circ - \tan 27^\circ - \tan 63^\circ + \tan 81^\circ$
5. Solve the equation $\sin \theta + \sin 3\theta + \sin 5\theta = 0$
6. Solve $2 \tan^2 x + \sec^2 x = 2$ for $0 \leq x \leq 2\pi$
7. Show that $2 \sin^2 \beta + 4 \cos(\alpha + \beta) \sin \alpha \sin \beta + \cos 2(\alpha + \beta) = \cos 2\alpha$
8. If $\sin(\theta + \alpha) = a$ and $\sin(\theta + \beta) = b$, then prove that $\cos 2(\alpha - \beta) - 4ab \cos(\alpha - \beta) = 1 - 2a^2 - 2b^2$

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ACUTE ANGLES AND RIGHT TRIANGLES



Acute angles are angles whose measurements fall between 0 and 90 degrees or has a radian of fewer than 90 degrees. When the term is given to a triangle as in an acute triangle, it means that all angles in the triangle are less than 90 degrees.

A right triangle is a triangle in which one of its interior angles is right angle (90°). The side opposite to the right angle is hypotenuse and sides adjacent to the right angle are called legs.

After studying this chapter, you will understand the core concepts of:

- Trigonometry of the Right Triangle
- Trigonometric Functions of an Acute Angle

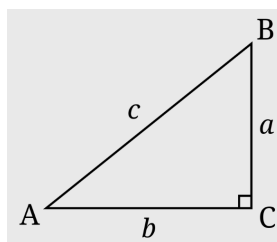
INTRODUCTION

An angle smaller than the right angle is called an acute angle. In other words, the angle which is less than 90 degrees forms an acute angle. The polygons such as triangle, parallelogram, trapezoid, etc. consist of at least one acute angle.

An acute angle is an angle that measures less than 90 degrees. In other words, it is an angle that is smaller than a right angle (which measures exactly 90 degrees) but larger than a zero-degree angle (which is a straight line). Acute angles are commonly encountered in geometry and trigonometry and are characterized by their sharpness and narrowness.

The two lines or line segments or rays with a common point leads to the formation of an angle. An angle is denoted through a symbol ‘ $\angle$ ’. An angle is measured with the help of a protractor in degrees ($^\circ$). An angle which is measuring less than 90 degrees is called an acute angle. This angle is smaller than the right angle (which is equal to 90 degrees). For example, $\angle 30^\circ$, $\angle 45^\circ$, $\angle 60^\circ$, $\angle 75^\circ$, $\angle 33^\circ$, $\angle 55^\circ$, $\angle 85^\circ$, etc. are all acute angles. The degree of an acute angle measures less than 90 degrees, i.e. less than a right angle. The examples of acute angle degrees are 12° , 35° , 48° , 65° , 80° , 89° . Hence, the acute angle degree ranges from 0 degrees and less than 90 degrees.

A right triangle is a triangle in which one of the angles is 90° . A 90-degree angle is called a right angle, and hence the triangle with a right angle is called a right triangle. Further, based on the other angle values, the right triangles are classified as isosceles right triangles and scalene right triangles.



The side opposite to the right angle is called the hypotenuse (side c in the figure). The sides adjacent to the right angle are called legs. Side a may be identified as the side adjacent to angle B and opposed to (or opposite) angle A , while side b is the side adjacent to angle A and opposed to angle B .

Sides

The three sides of a right triangle are related by the Pythagorean theorem, which in modern algebraic notation can be written

$$a^2 + b^2 = c^2,$$

where c is the length of the hypotenuse (side opposite the right angle), and a and b are the lengths of the legs (remaining two sides). Pythagorean triples are integer values of a , b , c satisfying this equation. This theorem was proven in antiquity, and is proposition I.47 in Euclid's Elements: "In right-angled triangles the square on

the side subtending the right angle is equal to the squares on the sides containing the right angle.”

Area

As with any triangle, the area is equal to one half the base multiplied by the corresponding height. In a right triangle, if one leg is taken as the base then the other is height, so the area of a right triangle is one half the product of the two legs. As a formula the area T is

$$T = \frac{1}{2}ab$$

where a and b are the legs of the triangle.

If the incircle is tangent to the hypotenuse AB at point P , then denoting the semi-perimeter $(a + b + c) / 2$ as s , we have $PA = s - a$ and $PB = s - b$, and the area is given by

$$T = PA \cdot PB = (s - a)(s - b).$$

This formula only applies to right triangles.

Altitudes

If an altitude is drawn from the vertex with the right angle to the hypotenuse then the triangle is divided into two smaller triangles which are both similar to the original and therefore similar to each other. From this:

- The altitude to the hypotenuse is the geometric mean (mean proportional) of the two segments of the hypotenuse.
- Each leg of the triangle is the mean proportional of the hypotenuse and the segment of the hypotenuse that is adjacent to the leg.

Inradius and Circumradius

The radius of the incircle of a right triangle with legs a and b and hypotenuse c is

$$r = \frac{a + b - c}{2} = \frac{ab}{a + b + c}.$$

The radius of the circumcircle is half the length of the hypotenuse,

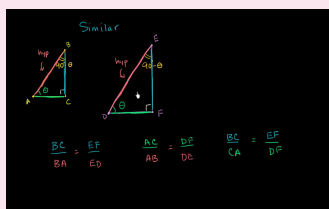
$$R = \frac{c}{2}.$$

Thus the sum of the circumradius and the inradius is half the sum of the legs:

$$R + r = \frac{a + b}{2}.$$

One of the legs can be expressed in terms of the inradius and the other leg as

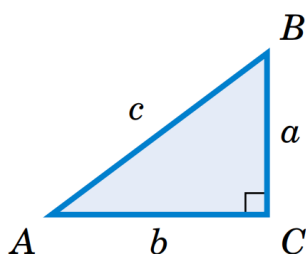
$$a = \frac{2r(b - r)}{1 - \frac{r}{b}}.$$



CORE CONCEPT: TRIGONOMETRY OF THE RIGHT TRIANGLE

Trigonometry is the study of the relations between the sides and angles of triangles. The word “trigonometry” is derived from the Greek words trigono, meaning “triangle”, and metro, meaning “measure”. Trigonometry is distinguished from elementary geometry in part by its extensive use of certain functions of angles, known as the trigonometric functions. Before discussing those functions, we will review some basic terminology about angles.

In a right triangle, the side opposite the right angle is called the hypotenuse, and the other two sides are called its legs. For example, in Figure the right angle is C, the hypotenuse is the line segment AB, which has length c , and BC and AC are the legs, with lengths a and b , respectively. The **hypotenuse** is always the longest side of a right triangle.



The Pythagorean Theorem

The solutions of many problems require the measurement of line segments and angles. When we use a ruler or tape measure to determine the length of a segment, or a protractor to find the measure of an angle, we are taking a direct measurement of the segment or the angle. In many situations, however, it is inconvenient or impossible to make a measurement directly. For example, it is difficult to make the direct measurements needed to answer the following questions:

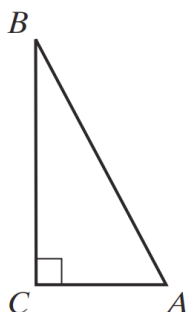
- What is the height of a 100-year-old oak tree?
- What is the width of a river?
- What is the distance to the sun?

We can answer these questions by using methods that involve indirect measurement. Starting with some known lengths of segments or angle measures, we apply a formula or a mathematical relationship to indirectly find the measurement in question. Engineers, surveyors, physicists, and astronomers frequently use these trigonometric

methods in their work. The figure at the left represents a right triangle. Recall that such a triangle contains one and only one right angle. In right triangle ABC, side $\overline{AB}$, which is opposite the right angle, is called the hypotenuse.

The hypotenuse is the longest side of the triangle. The other two sides of the triangle, $\overline{BC}$ and $\overline{AC}$, form the right angle. They are called the legs of the **right triangle**.

Right angle is an angle of exactly 90 degrees or radians corresponding to a quarter turn.



More than 2,000 years ago, the Greek mathematician Pythagoras demonstrated the following property of the right triangle, which is called the Pythagorean Theorem:

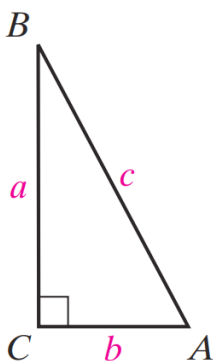
In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

If we represent the length of the hypotenuse of right triangle ABC by c and the lengths of the other two sides by a and b , the Theorem of Pythagoras may be written as the following formula:

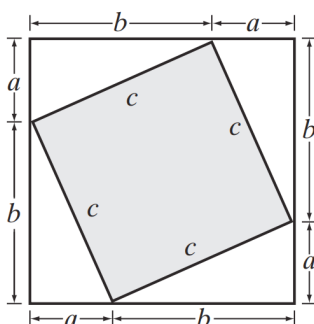
$$c^2 = a^2 + b^2$$

To show that this relationship is true for any right triangle ABC with the length of the hypotenuse represented by c and the lengths of the legs represented by a and b , consider a square with sides $(a + b)$.

You cannot have more than one 90 degree angle in a triangle, or it would be a rectangle.



The area of the square is $(a+b)^2$. However, since it is divided into four triangles and one smaller square, its area can also be expressed as:



$$\begin{aligned}\text{Area of the square} &= \text{area of the four triangles} \\ &\quad + \text{area of the smaller square} \\ &= 4\left(\frac{1}{2}\right)ab + c^2\end{aligned}$$

Although the area is written in two different ways, both expressions are equal.

Thus,

$$(a + b)^2 = 4\left(\frac{1}{2}\right)ab + c^2.$$

If we simplify, we obtain the relationship of the Pythagorean Theorem:

$$(a + b)^2 = 4\left(\frac{1}{2}\right)ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2 \text{ Expand the binomial term } (a+b)^2.$$

$$a^2 + 2ab - 2ab + b^2 = 2ab - 2ab + c^2 \text{ Subtract } 2ab \text{ from both sides of the equality.}$$

$$a^2 + b^2 = c^2$$

Statements of the Pythagorean Theorem

Two statements can be made for any right triangle where c represents the length of the hypotenuse (the longest side) and a and b represent the lengths of the other two sides.

1. If a triangle is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. If a triangle is a right triangle, then $c^2 = a^2 + b^2$.
2. If the square of the length of the longest side of a triangle is equal to the sum of the squares of the lengths of the other two sides, the triangle is a right triangle. If $c^2 = a^2 + b^2$ in a triangle, then the triangle is a right triangle.

If we know the lengths of any two sides of a right triangle, we can find the length of the third side. For example, if the measures of the legs of a right triangle are 7 and 9, we can write:

$$c^2 = a^2 + b^2$$

$$c^2 = 7^2 + 9^2$$

$$c^2 = 49 + 81$$

$$c^2 = 130$$

To solve this equation for c , we must do the opposite of squaring, that is, we must find the square root of 130. There are two square roots of 130, $+\sqrt{130}$ and $-\sqrt{130}$ which we write as $\pm\sqrt{130}$.

There are two things that we must consider here when finding the value of c .

1. Since c represents the length of a line segment, only the positive number is an acceptable value. Therefore, $c = +\sqrt{130}$.
2. There is no rational number that has a square of 130. The value of c is an **irrational number**. However, we usually use a calculator to find a rational approximation for the irrational number.

ENTER: **2nd** **$\sqrt{}$** 130 **ENTER**

DISPLAY: $\sqrt{130}$ 11.40175425

Irrational numbers are all the real numbers that are not rational numbers. That is, irrational numbers cannot be expressed as the ratio of two integers.

Therefore, to the nearest tenth, the length of the hypotenuse is 11.4. Note that the calculator gives only the positive rational approximation of the square root of 130.

Example

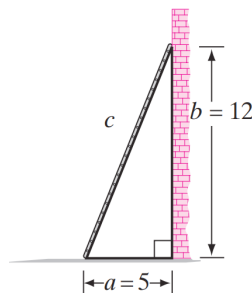
A ladder is placed 5 feet from the foot of a wall. The top of the ladder reaches a point 12 feet above the ground. Find the length of the ladder.

Solution

The ladder, the wall, and the ground form a right triangle. The length of the ladder is c , the length of the hypotenuse of the right triangle. The distance from the foot of the ladder to the wall is $a = 5$, and the distance from the ground to the top of the ladder is $b = 12$. Use the Theorem of Pythagoras.

Rational number is any number that can be expressed as the quotient or fraction p/q of two integers, a numerator p and a non-zero denominator q .

$$\begin{aligned}c^2 &= a^2 + b^2 \\c^2 &= 5^2 + 12^2 \\c^2 &= 25 + 144 \\c^2 &= 169 \\c &= \pm\sqrt{169} = \pm 13\end{aligned}$$



Reject the negative value. Note that in this case the exact value of c is a **rational number** because 169 is a perfect square.

Answer The length of the ladder is 13 feet.

Example

The hypotenuse of a right triangle is 36.0 centimeters long and one leg is 28.5 centimeters long.

- Find the length of the other leg to the nearest tenth of a centimeter.
- Find the area of the triangle using the correct number of significant digits.

Solution

- The length of the hypotenuse is $c = 36.0$ and the length of one leg is $a = 28.5$. The length of the other leg is b . Substitute the known values in the Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

$$36.0^2 = 28.5^2 + b^2$$

$$1,296 = 812.25 + b^2$$

$$483.75 = b^2$$

$$\pm\sqrt{483.75} = b$$

Reject the negative value. Use a calculator to find a rational approximation of the value of b . A calculator displays 21.99431745. Round the answer to the nearest tenth.

Answer: The length of the other leg is 22.0 centimeters.

b. Area of $\triangle ABC = \frac{1}{2}bh = \frac{1}{2}(28.5)(22.0) = 313.5$

Since the lengths are given to three significant digits, we will round the area to three significant digits.

Answer: The area of $\triangle ABC$ is 314 square centimeters.

Example

Is a triangle whose sides measure 8 centimeters, 7 centimeters, and 4 centimeters a right triangle?

Solution

If the triangle is a right triangle, the longest side, whose measure is 8, must be the hypotenuse. Then:

$$c^2 = a^2 + b^2$$

$$8^2 \stackrel{?}{=} 7^2 + 4^2$$

$$64 \stackrel{?}{=} 49 + 16$$

$$64 \neq 65$$

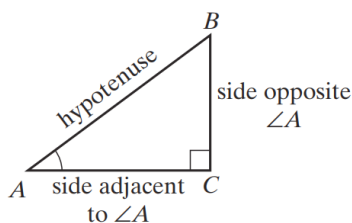
Answer: The triangle is not a right triangle.

The Tangent Ratio

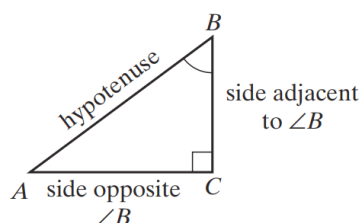
Naming Sides

In a right triangle, the hypotenuse, which is the longest side, is opposite the right angle. The other two sides in a right triangle are called the legs. However, in trigonometry of the right triangle, we call these legs the opposite side and the adjacent side to describe their relationship to one of the acute angles in the triangle.

Notice that $\triangle ABC$ is the same right triangle in both figures below, but the position names we apply to the legs change with respect to the angles.



In $\triangle ABC$:
 $\overline{BC}$ is opposite $\angle A$;
 $\overline{AC}$ is adjacent to $\angle A$.



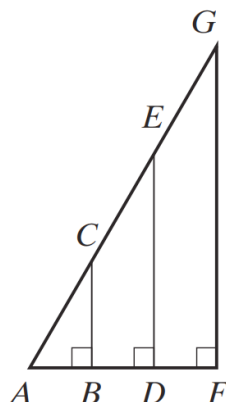
In $\triangle ABC$:
 $\overline{AC}$ is opposite $\angle B$;
 $\overline{BC}$ is adjacent to $\angle B$.

Similar Triangles

Three right triangles are drawn to coincide at vertex A. Since each triangle contains a right angle as well as $\angle A$, we know that the third angles of each triangle are congruent. When three angles of one triangle are congruent to the three angles of another, the triangles are similar.

The corresponding sides of similar triangles are in proportion. Therefore:

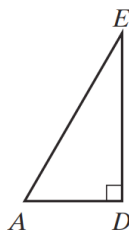
$$\frac{CB}{BA} = \frac{ED}{DA} = \frac{GF}{FA}$$



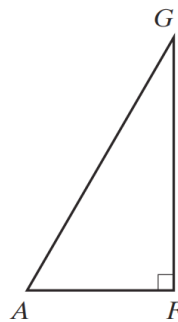
The similar triangles shown in the previous page, $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$, are separated and shown below.



In $\triangle ABC$:
 $\overline{CB}$ is opposite $\angle A$;
 $\overline{BA}$ is adjacent to $\angle A$.



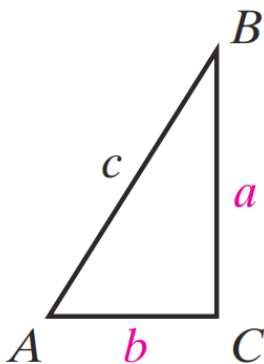
In $\triangle ADE$:
 $\overline{ED}$ is opposite $\angle A$;
 $\overline{DA}$ is adjacent to $\angle A$.



In $\triangle AFG$:
 $\overline{GF}$ is opposite $\angle A$;
 $\overline{FA}$ is adjacent to $\angle A$.

Therefore, $\frac{CB}{BA} = \frac{ED}{DA} = \frac{GF}{FA} = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} =$ a constant for $\triangle ABC$, $\triangle ADE$, $\triangle AFG$ and for any right triangle similar to these triangles, that is, for any right triangle with an acute angle congruent to $\angle A$. This ratio is called the tangent of the angle.

The tangent of an acute angle of a right triangle is the ratio of the length of the side opposite the acute angle to the length of the side adjacent to the acute angle.



For right triangle ABC, with $m\angle C = 90^\circ$, the definition of the tangent of $\angle A$ is as follows:

$$\text{tangent } A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent } \angle A} = \frac{BC}{AC} = \frac{a}{b}$$

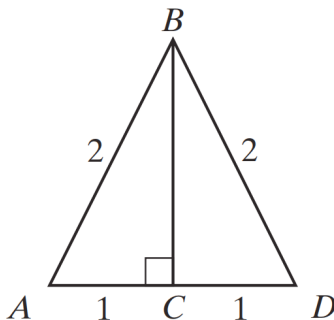
By using “tan A” as an abbreviation for tangent A, “opp” as an abbreviation for the length of the leg opposite $\angle A$, and “adj” as an abbreviation for the length of the leg adjacent to $\angle A$, we can shorten the way we write the relationship given above as follows:

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{a}{b}$$

Finding Tangent Ratios on a Calculator

The length of each side of equilateral triangle ABD is 2. The altitude $\overline{BC}$ from B to $\overline{AD}$ forms two congruent right triangles with $AC = 1$. We can use the hypotenuse rule to find BC.

$$\begin{aligned}(AC)^2 + (BC)^2 &= (AB)^2 \\ 1^2 + (BC)^2 &= 2^2 \\ 1 + (BC)^2 &= 4 \\ (BC)^2 &= 3 \\ BC &= \sqrt{3}\end{aligned}$$



The measure of each angle of an equilateral triangle is 60° . Therefore we can use the lengths of $\overline{AC}$ and $\overline{BC}$ to find the exact value of the tangent of a 60° angle.

$$\tan 60^\circ = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

But how can we find the constant value of the tangent ratio when the right triangle has an angle of 40° or 76° ? Since we want to work with the value of this ratio for any right triangle, no matter what the measures of the acute angles may be, mathematicians have compiled tables of the tangent values for angles with measures from 0° to 90° . Also, a calculator has the ability to display the value of this ratio for any angle. We will use a calculator to determine these values.

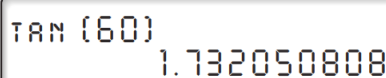
CASE 1

Given an angle measure, find the tangent ratio.

We saw that $\tan 60^\circ$ is equal to $\sqrt{3}$. The calculator will display this value as an approximate decimal. To find $\tan 60^\circ$, enter the sequence of keys shown below.

ENTER: **TAN** 60 **)** **ENTER**

DISPLAY:



TAN (60)
1.732050808

The value given in the calculator display is the rational approximation of $\sqrt{3}$, the value of $\tan 60^\circ$ that we found using the ratio of the lengths of the legs of a right triangle with a 60° angle. Therefore, to the nearest ten-thousandth, $\tan 60^\circ = 1.7321$.

CASE 2

Given a tangent ratio, find the angle measure.

The value of the tangent ratio is different for each different angle measure from 0° to 90° . Therefore, if we know the value of the tangent ratio, we can find the measure of the acute angle that has this tangent ratio. The calculator key used to do this is labeled **TAN⁻¹** and is accessed by first pressing **2nd**. We can think of $\tan^{-1}$ as “the angle whose tangent is.” Therefore, $\tan^{-1}(0.9004)$ can be read as “the angle whose tangent is 0.9004.”

To find the measure of $\angle A$ from the calculator, we use the following sequences of keys.

ENTER: **2nd** **TAN⁻¹** 0.9004 **)** **ENTER**

DISPLAY:

TAN⁻¹ (0.9004)
41.99987203

The measure of $\angle A$ to the nearest degree is 42° .

Example

In $\triangle ABC$, $\angle C$ is a right angle, $BC = 3$, $AC = 4$, and $AB = 5$.

a. Find:

- (1) $\tan A$
- (2) $\tan B$
- (3) $m\angle A$ to the nearest degree
- (4) $m\angle B$ to the nearest degree

In a right triangle, the tangents of the two acute angles are reciprocals.

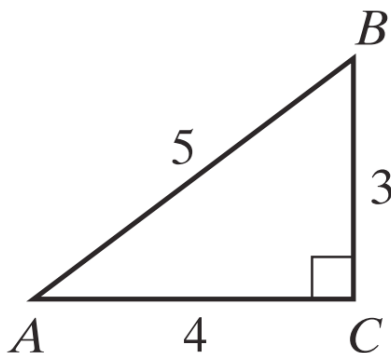
b. Show that the acute angles of the triangle are complementary.

Solution

$$(1) \tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{3}{4} \text{ Answer}$$

a.

$$(2) \tan B = \frac{\text{opp}}{\text{adj}} = \frac{AC}{BC} = \frac{4}{3} \text{ Answer}$$



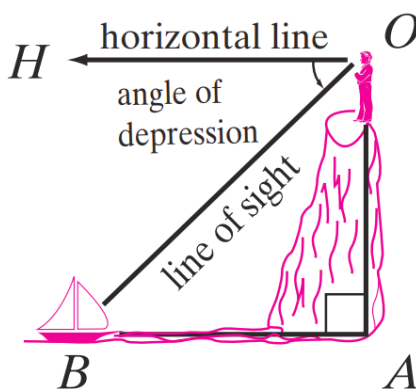
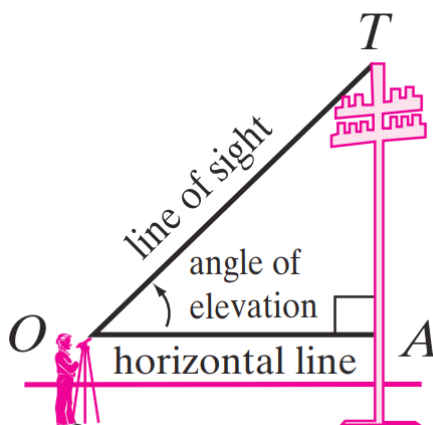
b. $m\angle A + m\angle B = 36.869889765^\circ + 53.13010235^\circ = 90.000000^\circ$. Therefore, the acute angles of $\triangle ABC$ are complementary. Answer

Applications of the Tangent Ratio

The tangent ratio is often used to make indirect measurements when the measures of a leg and an acute angle of a right triangle are known.

Angle of Elevation and Angle of Depression

When a telescope or some similar instrument is used to sight the top of a telephone pole, the instrument is elevated (tilted upward) from a horizontal position. Here, $\overrightarrow{OT}$ is the line of sight and $\overrightarrow{OA}$ is the horizontal line. The angle of elevation is the angle determined by the rays that are parts of the horizontal line and the line of sight when looking upward. Here, $\angle TOA$ is the angle of elevation.



When an instrument is used to sight a boat from a cliff, the instrument is depressed (tilted downward) from a horizontal position. Here, $\overrightarrow{OB}$ is the line of sight and $\overrightarrow{OH}$ is the horizontal line. The angle of depression is the angle determined by the rays that are parts of the horizontal line and of the line of sight when looking downward. Here, $\angle HOB$ is the angle of depression.

Note that, if $\overleftrightarrow{BA}$ is a horizontal line and $\overleftrightarrow{BO}$ is the line of sight from the boat to the top of the cliff, $\angle ABO$ is called the angle of elevation from the boat to the top of the

cliff. Since $\vec{HO} \parallel \vec{BA}$ and $\vec{OB}$ is a transversal, alternate interior angles are congruent, namely, $\angle HOB \cong \angle ABO$. Thus, the angle of elevation measured from B to O is congruent to the angle of depression measured from O to B.

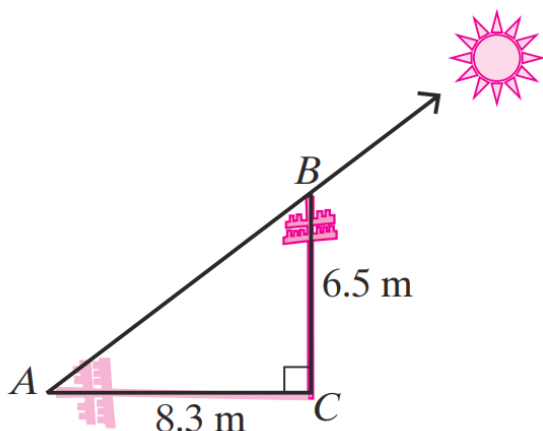
Example

Find to the nearest degree the measure of the angle of elevation of the sun when a vertical pole 6.5 meters high casts a shadow 8.3 meters long.

Solution

The angle of elevation of the sun is the same as $\angle A$, the angle of elevation to the top of the pole from A, the tip of the shadow. Since the vertical pole and the shadow are the legs of a right triangle opposite and adjacent to $\angle A$, use the **tangent ratio**.

Tangent ratio is a tool used with right triangles that allows one to find the length of the sides of a triangle given the degree of its angles.



$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{6.5}{8.3}$$

Answer: To the nearest degree, the measure of the angle of elevation of the sun is 38° .

Example

At a point on the ground 39 meters from the foot of a tree, the measure of the angle of elevation of the top of the tree is 42° . Find the height of the tree to the nearest meter.

Solution

Let T be the top of the tree, A be the foot of the tree, and B be the point on the ground 39 meters from A. Draw $\triangle ABT$, and label the diagram: $m\angle B 42^\circ$, AB 39.

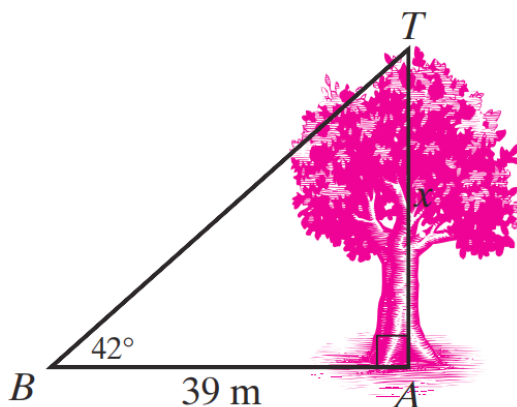
Let x = height of tree (AT). The height of the tree is the length of the perpendicular from the top of the tree to the ground. Since the problem involves the measure of an acute angle and the measures of the legs of a right triangle, use the tangent ratio:

$$\tan B = \frac{\text{opp}}{\text{adj}}$$

$$\tan B = \frac{AT}{BA}$$

$$\tan 42^\circ = \frac{x}{39} \text{ Substitute the given values.}$$

$$x = 39 \tan 42^\circ \text{ Solve for } x.$$



Use a calculator for the computation: $x = 35.11575773$

Answer: To the nearest meter, the height of the tree is 35 meters.

Example

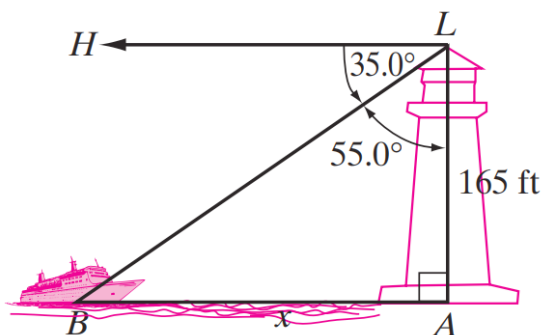
From the top of a lighthouse 165 feet above sea level, the measure of the angle of depression of a boat at sea is 35.0° . Find to the nearest foot the distance from the boat to the foot of the lighthouse.

Solution

Let L be the top of the lighthouse, LA be the length of the perpendicular from L to sea level, and B be the position of the boat. Draw right triangle ABL, and draw $\overrightarrow{LH}$, the horizontal line through L.

Since $\angle HLB$ is the angle of depression, $m\angle HLB = 35.0$, $m\angle LBA = 35.0$, and $m\angle BLA = 90 - 35.0 = 55.0$.

Let x = distance from the boat to the foot of the lighthouse (BA).



Method 1

Using $\angle BLA$, $\overline{BA}$ is the opposite side and $\overline{LA}$ is the adjacent side. Form the tangent ratio:

$$\tan \angle BLA = \frac{\overline{BA}}{\overline{LA}}$$

$$\tan 55.0^\circ = \frac{x}{165}$$

$$x = 165 \tan 55.0^\circ$$

Use a calculator to perform the computation. The display will read 235.644421.

Method 2

Using $\angle LBA$, $\overline{LA}$ is the opposite side and $\overline{BA}$ is the adjacent side. Form the tangent ratio:

$$\tan \angle LBA = \frac{\overline{LA}}{\overline{BA}}$$

$$\tan 35.0^\circ = \frac{165}{x}$$

$$x \tan 35.0^\circ = 165$$

$$x = \frac{165}{\tan 35.0^\circ}$$

Use a calculator to perform the computation. The display will read 235.644421.

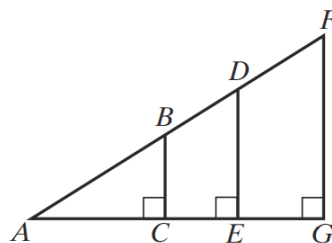
Answer: To the nearest foot, the boat is 236 feet from the foot of the lighthouse.

The Sine and Cosine Ratios

Since the tangent is the ratio of the lengths of the two legs of a right triangle, it is not directly useful in solving problems in which the hypotenuse is involved. In trigonometry of the right triangle, two ratios that involve the hypotenuse are called the sine of an angle and the cosine of an angle.

As in our discussion of the tangent ratio, we recognize that the figure at the right shows three similar triangles. Therefore, the ratios of corresponding sides are equal.

Acute angle is an angle smaller than a right angle (it is less than 90 degrees).



The Sine Ratio

From the figure, we see that

$$\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF} = \text{a constant}$$

This ratio is called the sine of $\angle A$.

The sine of an acute angle of a right triangle is the ratio of the length of the side opposite the **acute angle** to the length of the hypotenuse.

In right triangle ABC, with $m\angle C = 90^\circ$, the definition of the sine of $\angle A$ is:

$$\sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{BC}{AB} = \frac{a}{c}$$

By using “sin A” as an abbreviation for sine A, “opp” as an abbreviation for the length of the leg opposite $\angle A$, and “hyp” as an abbreviation for the length of the hypotenuse, we can shorten the way we write the definition of sine A as follows:

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{a}{c}$$

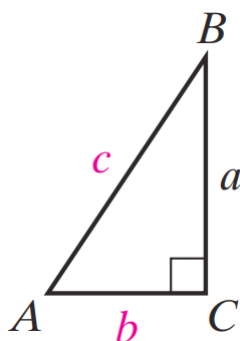
The Cosine Ratio

From the preceding figure, which shows similar triangles, $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$, we see that

$$\frac{AC}{AB} = \frac{AE}{AD} = \frac{AG}{AF} = \text{a constant.}$$

This ratio is called the cosine of $\angle A$.

The cosine of an acute angle of a right triangle is the ratio of the length of the side adjacent to the acute angle to the length of the hypotenuse.

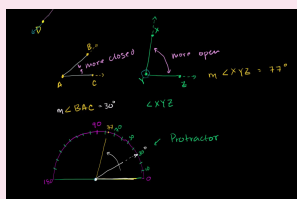


In right triangle ABC, with $m\angle C = 90^\circ$, the definition of the cosine of $\angle A$ is:

$$\cos A = \frac{\text{length of side adjacent } \angle A}{\text{length of hypotenuse}} = \frac{AC}{AB} = \frac{b}{c}$$

By using “cos A” as an abbreviation for cosine A, “adj” as an abbreviation for the length of the leg adjacent to $\angle A$, and “hyp” as an abbreviation for the length of the hypotenuse, we can shorten the way we write the definition of cosine A as follows:

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{b}{c}$$



CORE CONCEPT: TRIGONOMETRIC FUNCTIONS OF AN ACUTE ANGLE

An acute angle (“acute” meaning “sharp”) is an angle smaller than a right angle (it is less than 90 degrees). A triangle in which one angle is 90° is called a right triangle. The side opposite to the right angle is called the hypotenuse and the remaining sides are called the legs of the triangle.

Consider a right triangle $\triangle ABC$, with the right angle at C and with lengths a, b, and c, as in the figure on the right. For the acute angle A, call the leg $\overline{BC}$ its opposite side, and call the leg $\overline{AC}$ its adjacent side. Recall that the hypotenuse of the triangle is the side $\overline{AB}$. The ratios of sides of a right triangle occur often enough in practical applications to warrant their own names, so we define the six trigonometric functions of A as follows:

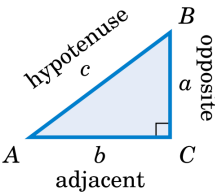


Table 1: The six trigonometric functions of A

Name of function	Abbreviation	Definition
sine A	$\sin A$	$= \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}$
cosine A	$\cos A$	$= \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}$
tangent A	$\tan A$	$= \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}$
cosecant A	$\csc A$	$= \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$
secant A	$\sec A$	$= \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$
cotangent A	$\cot A$	$= \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$

We will usually use the abbreviated names of the functions. Notice from Table 1 that the pairs $\sin A$ and $\csc A$, $\cos A$ and $\sec A$, and $\tan A$ and $\cot A$ are reciprocals:

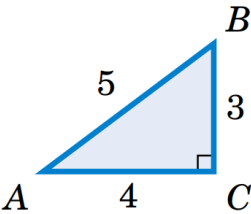
$$\csc A = \frac{1}{\sin A} \qquad \sec A = \frac{1}{\cos A} \qquad \cot A = \frac{1}{\tan A}$$
$$\sin A = \frac{1}{\csc A} \qquad \cos A = \frac{1}{\sec A} \qquad \tan A = \frac{1}{\cot A}$$

Example

For the right triangle $\triangle ABC$ shown on the right, find the values of all six trigonometric functions of the acute angles A and B.

Solution

The hypotenuse of $\triangle ABC$ has length 5. For angle A, the opposite side $\overline{BC}$ has length 3 and the adjacent side $\overline{AC}$ has length 4. Thus:



$$\begin{aligned}\sin A &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} & \cos A &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} & \tan A &= \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} \\ \csc A &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} & \sec A &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} & \cot A &= \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3}\end{aligned}$$

For angle B, the opposite side $\overline{AC}$ has length 4 and the adjacent side $\overline{BC}$ has length 3. Thus:

$$\begin{aligned}\sin B &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} & \cos B &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} & \tan B &= \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \\ \csc B &= \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4} & \sec B &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3} & \cot B &= \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}\end{aligned}$$

Example that we did not specify the units for the lengths. This raises the possibility that our answers depended on a triangle of a specific physical size.

For example, suppose that two different students are reading this textbook: one in the United States and one in Germany. The American student thinks that the lengths 3, 4, and 5 in Example are measured in inches, while the German student thinks that they are measured in centimeters. Since 1 in ≈ 2.54 cm, the students are using triangles of different physical sizes (see Figure 1 below, not drawn to scale).

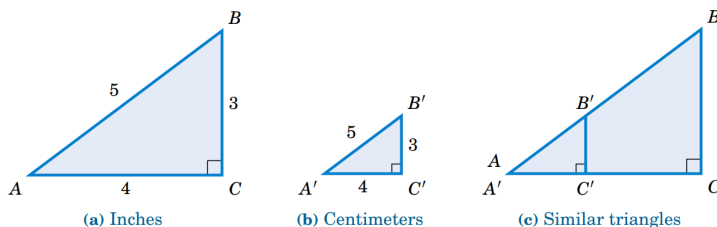


Figure 1: $\triangle ABC \sim \triangle A'B'C'$.

If the American triangle is $\triangle ABC$ and the German triangle is $\triangle A'B'C'$, then we see from Figure 1 that $\triangle ABC$ is similar to $\triangle A'B'C'$, and hence the corresponding angles are equal and the ratios of the corresponding sides are equal. In fact, we know that common ratio: the sides of $\triangle ABC$ are approximately 2.54 times longer than the corresponding sides of $\triangle A'B'C'$. So when the American student calculates $\sin A$ and the German student calculates $\sin A'$, they get the same answer: 3

$$\triangle ABC \sim \triangle A'B'C' \Rightarrow \frac{BC}{B'C'} = \frac{AB}{A'B'} \Rightarrow \frac{BC}{AB} = \frac{B'C'}{A'B'} \Rightarrow \sin A = \sin A'$$

Likewise, the other values of the trigonometric functions of A and A' are the same. In fact, our argument was general enough to work with any similar right triangles. This leads us to the following conclusion:

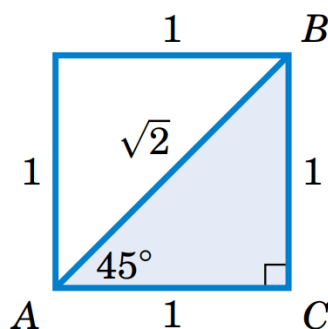
When calculating the trigonometric functions of an acute angle A, you may use any right triangle which has A as one of the angles.

isosceles triangle is a triangle that has two sides of equal length. Sometimes it is specified as having exactly two sides of equal length, and sometimes as having at least two sides of equal length, the latter version thus including the equilateral triangle as a special case.

Since we defined the trigonometric functions in terms of ratios of sides, you can think of the units of measurement for those sides as canceling out in those ratios. This means that the values of the trigonometric functions are unitless numbers. So when the American student calculated $3/5$ as the value of $\sin A$ in Example, that is the same as the $3/5$ that the German student calculated, despite the different units for the lengths of the sides.

Example

Find the values of all six trigonometric functions of 45° .



Solution

Since we may use any right triangle which has 45° as one of the angles, use the simplest one: take a square whose sides are all 1 unit long and divide it in half diagonally, as in the figure on the right. Since the two legs of the triangle $\triangle ABC$ have the same length, $\triangle ABC$ is an **isosceles triangle**, which means that the angles A and B are equal. So since $A + B = 90^\circ$, this means that we must have $A = B = 45^\circ$. By the Pythagorean Theorem, the length c of the hypotenuse is given by

$$c^2 = 1^2 + 1^2 = 2 \Rightarrow c = \sqrt{2}.$$

Thus, using the angle A we get:

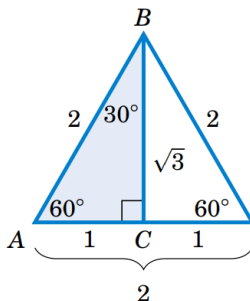
$$\begin{aligned} \sin 45^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} & \cos 45^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} & \tan 45^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \\ \csc 45^\circ &= \frac{\text{hypotenuse}}{\text{opposite}} = \sqrt{2} & \sec 45^\circ &= \frac{\text{hypotenuse}}{\text{adjacent}} = \sqrt{2} & \cot 45^\circ &= \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{1} = 1 \end{aligned}$$

Note that we would have obtained the same answers if we had used any right triangle similar to $\triangle ABC$. For example, if we multiply each side of $\triangle ABC$ by $\sqrt{2}$, then we would have a similar triangle with legs of length $\sqrt{2}$ and hypotenuse of

length 2. This would give us $\sin 45^\circ = \frac{\sqrt{2}}{2}$, which equals $\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2} \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}$ as before. The same goes for the other functions.

Example

Find the values of all six trigonometric functions of 60° .



Solution

Since we may use any right triangle which has 60° as one of the angles, we will use a simple one: take a triangle whose sides are all 2 units long and divide it in half by drawing the bisector from one vertex to the opposite side, as in the figure on the right. Since the original triangle was an equilateral triangle (i.e. all three sides had the same length), its three angles were all the same, namely 60° . Recall from elementary geometry that the bisector from the vertex angle of an equilateral triangle to its opposite side bisects both the vertex angle and the opposite side. So as in the figure on the right, the triangle $\triangle ABC$ has angle $A = 60^\circ$ and angle $B = 30^\circ$

A 45-45-90 triangle is what you get when you have an isosceles right triangle. If a triangle is a right triangle and it's isosceles, then the two sides other than the right angle must add up to 90 degrees and they must be equal, so each must equal 45 degrees.

, which forces the angle C to be 90° . Thus, $\triangle ABC$ is a right triangle. We see that the hypotenuse has length $c = AB = 2$ and the leg $\overline{AC}$ has length $b = AC = 1$. By the Pythagorean Theorem, the length a of the leg $\overline{BC}$ is given by

$$a^2 + b^2 = c^2 \Rightarrow a^2 = 2^2 - 1^2 = 3 \Rightarrow a = \sqrt{3}.$$

Thus, using the angle A we get:

$$\sin 60^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \quad \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \quad \tan 60^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

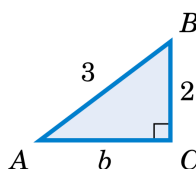
$$\csc 60^\circ = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{2}{\sqrt{3}} \quad \sec 60^\circ = \frac{\text{hypotenuse}}{\text{adjacent}} = 2 \quad \cot 60^\circ = \frac{\text{adjacent}}{\text{opposite}} = \frac{1}{\sqrt{3}}$$

Notice that, as a bonus, we get the values of all six trigonometric functions of 30° , by using angle $B = 30^\circ$ in the same triangle $\triangle ABC$ above:

$$\begin{aligned}\sin 30^\circ &= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} & \cos 30^\circ &= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} & \tan 30^\circ &= \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \\ \csc 30^\circ &= \frac{\text{hypotenuse}}{\text{opposite}} = 2 & \sec 30^\circ &= \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{2}{\sqrt{3}} & \cot 30^\circ &= \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{3}}{1} = \sqrt{3}\end{aligned}$$

Example

A is an acute angle such that $\sin A = \frac{2}{3}$. Find the values of the other trigonometric functions of A.



Solution

In general it helps to draw a right triangle to solve problems of this type. The reason is that the trigonometric functions were defined in terms of ratios of sides of a right triangle, and you are given one such function (the sine, in this case) already in terms of a ratio: $\sin A = \frac{2}{3}$. Since $\sin A$ is defined as $\frac{\text{opposite}}{\text{hypotenuse}}$, use 2 as the length of the side opposite A and use 3 as the length of the hypotenuse in a right triangle $\triangle ABC$ (see the figure above), so that $\sin A = \frac{2}{3}$. The adjacent side to A has unknown length b , but we can use the Pythagorean Theorem to find it:

$$2^2 + b^2 = 3^2 \Rightarrow b^2 = 9 - 4 = 5 \Rightarrow b = \sqrt{5}$$

We now know the lengths of all sides of the triangle $\triangle ABC$, so we have:

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \quad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{5}}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{2} \quad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{\sqrt{5}} \quad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{5}}{2}$$

Cotangent is one of the 6 trigonometric functions. It is usually referred to as “cot”. Just like other trigonometric ratios, the cotangent formula is also defined as the ratio of the sides of a right-angled triangle.

You may have noticed the connections between the sine and cosine, secant and cosecant, and tangent and **cotangent** of the complementary angles in Examples. Generalizing those examples gives us the following theorem:

Theorem 1. Cofunction Theorem: If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold:

$$\sin A = \cos B$$

$$\sin A = \cos B$$

$$\tan A = \cot B$$

$$\sin B = \cos A$$

$$\sec B = \csc A$$

$$\tan B = \cot A$$

We say that the pairs of functions $\{\sin, \cos\}$, $\{\sec, \csc\}$, and $\{\tan, \cot\}$ are cofunctions.

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the “co” in their names. The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its cofunction of the complementary angle.

Example

Write each of the following numbers as trigonometric functions of an angle less than 45° : (a) $\sin 65^\circ$; (b) $\cos 78^\circ$; (c) $\tan 59^\circ$.

Solution

- (a) The complement of 65° is $90^\circ - 65^\circ = 25^\circ$ and the cofunction of $\sin$ is $\cos$, so by the Cofunction Theorem we know that $\sin 65^\circ = \cos 25^\circ$.
- (b) The complement of 78° is $90^\circ - 78^\circ = 12^\circ$ and the cofunction of $\cos$ is $\sin$, so $\cos 78^\circ = \sin 12^\circ$.
- (c) The complement of 59° is $90^\circ - 59^\circ = 31^\circ$ and the cofunction of $\tan$ is $\cot$, so $\tan 59^\circ = \cot 31^\circ$.

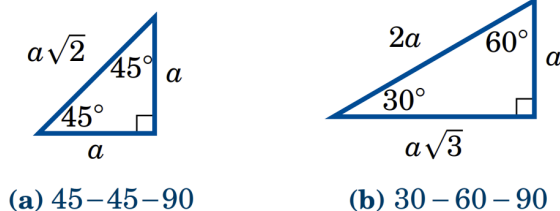
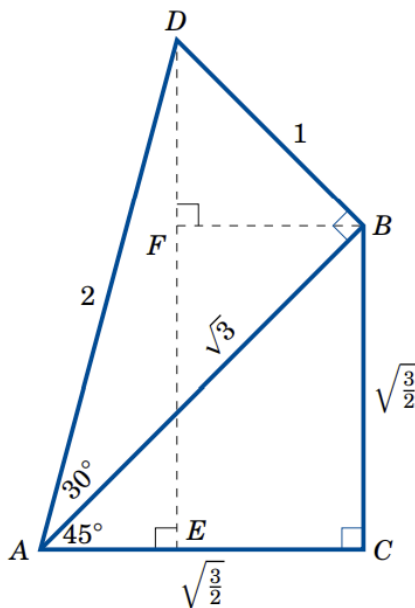


Figure 2: Two general right triangles (any $a > 0$).

The angles 30° , 45° , and 60° arise often in applications. We can use the Pythagorean Theorem to generalize the right triangles in Examples and see what any $45-45-90$ and $30-60-90$ right triangles look like, as in Figure 2 above.

Example

Find the sine, cosine, and tangent of 75° .



Solution

Since $75^\circ = 45^\circ + 30^\circ$, place a 30–60–90 right triangle $\triangle ADB$ with legs of length $\sqrt{3}$ and 1 on top of the hypotenuse of a 45–45–90 right triangle $\triangle ABC$ whose hypotenuse has length $\sqrt{3}$, as in the figure on the right. From Figure 2(a) we know that the length of each leg of $\triangle ABC$ is the length of the hypotenuse divided by $\sqrt{2}$.

So $AC = BC = \frac{\sqrt{3}}{\sqrt{2}} = \sqrt{\frac{3}{2}}$. Draw $\overline{DE}$ perpendicular to $\overline{AC}$, so that $\triangle ADE$ is a right triangle. Since $\angle BAC = 45^\circ$ and $\angle DAB = 30^\circ$, we see that $\angle DAE = 75^\circ$ since it is the sum of those two angles. Thus, we need to find the sine, cosine, and tangent of $\angle DAE$.

Notice that $\angle ADE = 15^\circ$, since it is the complement of $\angle DAE$. And $\angle ADB = 60^\circ$, since it is the complement of $\angle DAB$. Draw $\overline{BF}$ perpendicular to $\overline{DE}$, so that $\triangle DFB$ is a right triangle. Then $\angle BDF = 45^\circ$, since it is the difference of $\angle ADB = 60^\circ$ and $\angle ADE = 15^\circ$. Also, $\angle DBF = 45^\circ$ since it is the complement of $\angle BDF$. The hypotenuse $\overline{BD}$ of $\triangle DFB$ has length 1 and $\triangle DFB$ is a 45–45–90 right triangle, so we know that $DF = FB = \frac{1}{\sqrt{2}}$.

Now, we know that $\overline{DE} \perp \overline{AC}$ and $\overline{BC} \perp \overline{AC}$, so $\overline{FE}$ and $\overline{BC}$ are parallel. Likewise, $\overline{FB}$ and $\overline{EC}$ are both perpendicular to $\overline{DE}$ and hence $\overline{FB}$ is parallel to $\overline{EC}$. Thus, $FBCE$ is a rectangle, since $\angle BCE$ is a right angle. So $EC = FB = \frac{1}{\sqrt{2}}$ and $FE = BC = \sqrt{\frac{3}{2}}$. Hence,

$$DE = DF + FE = \frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}} = \frac{\sqrt{3} + 1}{\sqrt{2}}$$

and

$$AE = AC - EC = \sqrt{\frac{3}{2}} - \frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{2}}.$$

Thus,

$$\sin 75^\circ = \frac{DE}{AD} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4},$$

and

$$\tan 75^\circ = \frac{DE}{AE} = \frac{\frac{\sqrt{3}+1}{\sqrt{2}}}{\frac{\sqrt{3}-1}{\sqrt{2}}} = \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$$

Note: Taking reciprocals, we get $\csc$

$$75^\circ = \frac{4}{\sqrt{6}+\sqrt{2}}, \sec 75^\circ = \frac{4}{\sqrt{6}-\sqrt{2}}, \text{ and } \cot 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}.$$

Trigonometric Functions of Any Angle

Right triangles are useful when trying to calculate the trigonometric functions of acute angles. What about angles that are not acute angles? In this you will learn (1) of how to compute the trigonometric functions of any angle, not just acute angles, (2) the sign of the trigonometric functions in each quadrant of the coordinate plane, and (3) the use of reference angles which reduce the question of finding the trigonometric functions of an angle to that of finding the trigonometric functions of the special angles 30° , 45° , and 60° .

Let θ be an angle in standard position as shown in Figure 3.

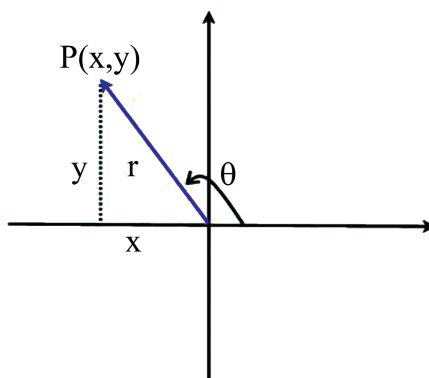


Figure 3.

Let $P(x, y)$ be any point on the terminal side. If r is the distance from the origin to the point P then by the Pythagorean Theorem, $r = \sqrt{x^2 + y^2}$. We define the trigonometric functions of θ to be

$$\begin{array}{lll} \sin \theta & = & \frac{y}{r} \\ \csc \theta & = & \frac{r}{y} \end{array} \quad \begin{array}{lll} \cos \theta & = & \frac{x}{r} \\ \sec \theta & = & \frac{r}{x} \end{array} \quad \begin{array}{lll} \tan \theta & = & \frac{y}{x} \\ \cot \theta & = & \frac{x}{y} \end{array}$$

where $x \neq 0$ and $y \neq 0$. If $\theta = k\pi$, where k is an integer, then the functions $\csc \theta$ and $\cot \theta$ are undefined since a point on the terminal side has components $P(x, 0)$.

Similarly, if $\theta = (2k + 1)\pi/2$ then the functions $\sec \theta$ and $\tan \theta$ are undefined since $P(0, y)$.

Do the above definitions depend on the choice of the point P? The answer is no. To see this, let $Q(x', y')$ be any other point on the terminal side of θ . See Figure 4.

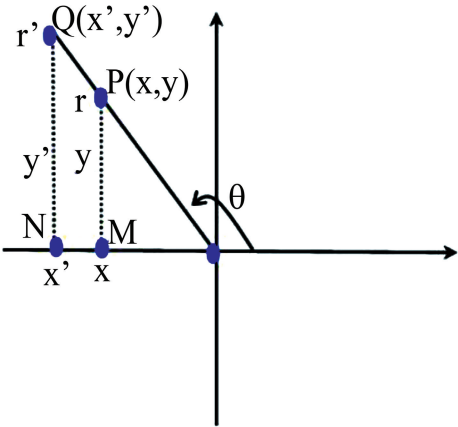


Figure 4.

Then the right triangles $\triangle OPM$ and $\triangle OQN$ are similar triangles since corresponding angles are equal. This implies that the ratios of the corresponding sides are equal. Thus

$$\frac{r}{r'} = \frac{x}{x'} = \frac{y}{y'}$$

or equivalently

$$\frac{x}{r} = \frac{x'}{r'} = \frac{y}{r} = \frac{y'}{r'} = \frac{y}{x} = \frac{y'}{x'}$$

Thus,

$$\begin{aligned} \sin \theta &= \frac{y'}{r'} & \cos \theta &= \frac{x'}{r'} & \tan \theta &= \frac{y'}{x'} \\ \csc \theta &= \frac{r'}{y'} & \sec \theta &= \frac{r'}{x'} & \cot \theta &= \frac{x'}{y'}. \end{aligned}$$

This shows that the trigonometric functions are independent of the point chosen on the terminal side of the angle.

Example

Complete the following chart, using the definitions introduced above.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
0°						
90°						
180°						
270°						

Solution

For $\theta = 0^\circ$ we choose the point $P(1, 0)$. For $\theta = 90^\circ$ we choose $P(0, 1)$; for $\theta = 180^\circ$, we choose $P(-1, 0)$ and finally for $\theta = 270^\circ$ we choose $P(0, -1)$. Then by the above definitions we have

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\csc \theta$	$\sec \theta$	$\cot \theta$
0°	0	1	0	undefined	1	undefined
90°	1	0	undefined	1	undefined	0
180°	0	-1	0	undefined	-1	undefined
270°	-1	0	undefined	-1	undefined	0

Example

Find the exact value of the six trigonometric functions of an angle θ if $P(4, -3)$ is a point on its terminal side.

Solution

First, note that $r = \sqrt{16 + 9} = 5$. Thus,

$$\sin \theta = \frac{-3}{5}; \cos \theta = \frac{4}{5}; \tan \theta = \frac{-3}{4}$$

$$\csc \theta = \frac{-5}{3}; \sec \theta = \frac{5}{4}; \cot \theta = \frac{-4}{3}.$$

Example

Given that $\cos \theta = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$, find the exact value of each of the remaining trigonometric functions.

Solution

Since θ is in Quadrant II, $\sin \theta > 0$, $\csc \theta > 0$, $\sec \theta < 0$, $\tan \theta < 0$, $\cot \theta < 0$. Thus, $x = -2$, $r = 3$ and by the Pythagorean formula $y = \sqrt{r^2 - x^2} = \sqrt{9 - 4} = \sqrt{5}$. It follows that $\sin \theta = \frac{\sqrt{5}}{3}$. So $\csc \theta = \frac{3\sqrt{5}}{5}$, $\tan \theta = -\frac{\sqrt{5}}{2}$, $\cot \theta = -\frac{2\sqrt{5}}{5}$, and finally $\sec \theta = -\frac{3}{2}$.

Example

Complete the following chart of signs of the six trigonometric functions.

Q	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
I						
II						
III						
IV						

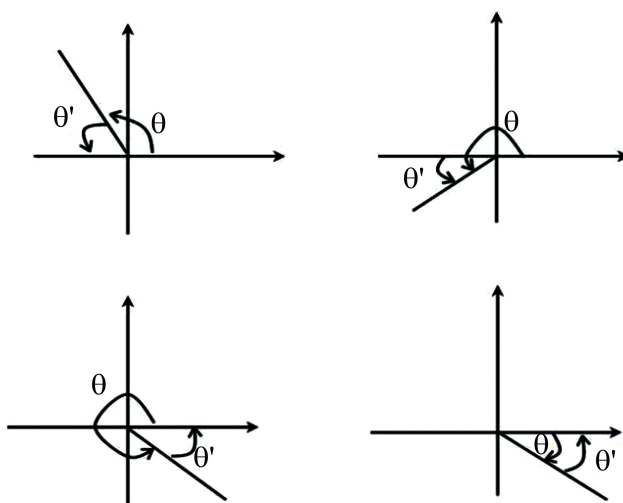
Solution

Q	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Reference Angles

The values of trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding angles called reference angles.

For an angle θ in standard position, the acute angle θ' between the terminal side of θ and either the positive or negative x-axis is called the reference angle of θ . Figure 5 illustrates the reference angles for some general angles θ .

**Figure 5.****Example**

Determine the reference angles for the following given angles:

- (a) -70° (b) 255° (c) $\frac{5\pi}{3}$ rad

Solution

- (a) $\theta' = 70^\circ$ (b) $\theta' = 75^\circ$ (c) $\theta' = \frac{\pi}{3}$.

Note that if θ is a non-acute angle and θ' is its reference angle then the trigonometric values of θ are equal to the trigonometric values of θ' with the appropriate sign which depends on the quadrant in which θ lies.

Example

Evaluate the following:

(a) $\cos \frac{4\pi}{3}$ (b) $\tan(-210^\circ)$ (c) $\csc \frac{11\pi}{3}$.

Solution

(a) $\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$.

(b) $\tan(-210^\circ) = -\tan 210^\circ = -\tan \frac{\pi}{6} = -\frac{\sqrt{3}}{3}$.

(c) $\csc \frac{11\pi}{3} = \csc(2\pi + \frac{3\pi}{4}) = \csc \frac{3\pi}{4} = \csc \frac{\pi}{4} = \sqrt{2}$.

Example

Referring to Figure 6, answer the following questions.

(a) Express the area of $\triangle OBC$ in terms of $\sin \theta$ and $\cos \theta$.

(b) Express the area of $\triangle OBD$ in terms of $\tan \theta$.

(c) Use parts (a) and (b) to show

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

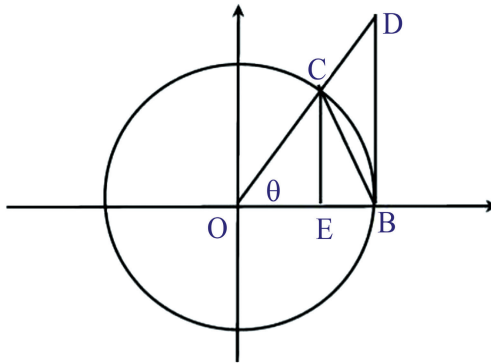


Figure 6.

Solution

(a) Area $\triangle OBC = \frac{1}{2}|EC||OB| = \frac{1}{2}|EC| = \frac{1}{2}\sin \theta$ since $\sin \theta = \frac{|EC|}{|OC|} = |EC|$.

(b) Area $\triangle OBD = \frac{1}{2}|BD||OB| = \frac{1}{2}|BD| = \frac{1}{2}\tan \theta$ since $\tan \theta = \frac{|DB|}{|OB|} = |DB|$.

(c) Using Figure 6 we see that

$$\text{Area } \triangle OBC < \text{Area circular sector OBC} < \text{Area } \triangle OBD.$$

But the area of the circular sector OBC is $\frac{1}{2}\theta$. Hence,

$$\frac{1}{2} \sin \theta < \frac{1}{2} \theta < \frac{1}{2} \tan \theta.$$

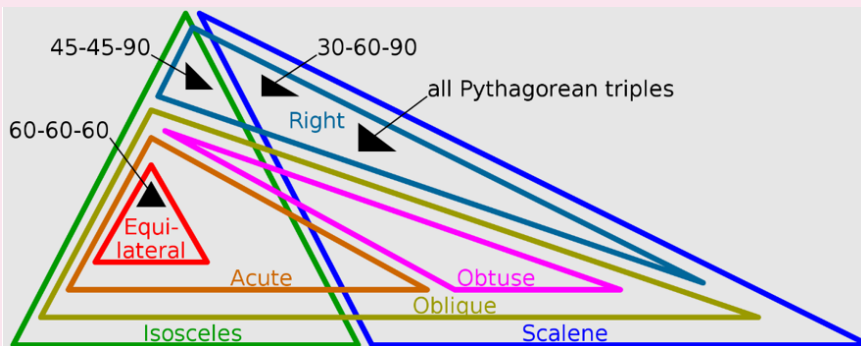
Multiplying through by $\frac{2}{\sin \theta}$ to obtain

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

Note that according to the given figure the angle is assumed to be in the first quadrant so that the sine function is positive there.

CONCEPT OF SPECIAL RIGHT TRIANGLE

A special right triangle is a right triangle with some regular feature that makes calculations on the triangle easier, or for which simple formulas exist. For example, a right triangle may have angles that form simple relationships, such as $45^\circ-45^\circ-90^\circ$. This is called an “angle-based” right triangle. A “side-based” right triangle is one in which the lengths of the sides form ratios of whole numbers, such as 3 : 4 : 5, or of other special numbers such as the golden ratio. Knowing the relationships of the angles or ratios of sides of these special right triangles allows one to quickly calculate various lengths in geometric problems without resorting to more advanced methods.



SUMMARY

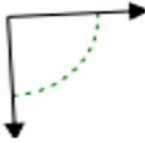
- An acute angle is an angle that measures less than 90 degrees. In other words, it is an angle that is smaller than a right angle (which measures exactly 90 degrees) but larger than a zero-degree angle (which is a straight line).
- A right triangle is a triangle in which one of the angles is 90° . A 90-degree angle is called a right angle, and hence the triangle with a right angle is called a right triangle.
- Two statements can be made for any right triangle where c represents the length of the hypotenuse (the longest side) and a and b represent the lengths of the other two sides.
- The value of the tangent ratio is different for each different angle measure from 0° to 90° .
- The tangent ratio is often used to make indirect measurements when the measures of a leg and an acute angle of a right triangle are known.
- Since the tangent is the ratio of the lengths of the two legs of a right triangle, it is not directly useful in solving problems in which the hypotenuse is involved.
- The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its co-function of the complementary angle.
- Right triangles are useful when trying to calculate the trigonometric functions of acute angles.
- The values of trigonometric functions of angles greater than 90° (or less than 0°) can be determined from their values at corresponding angles called reference angles.

MULTIPLE CHOICE QUESTIONS

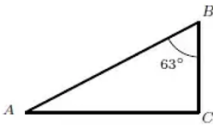
1. Classify each angle as acute, obtuse, right or straight



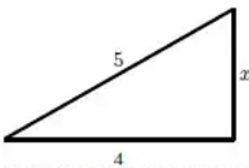
- a. Obtuse angle
 - b. Acute angle
 - c. Right angle
 - d. Straight angle
2. Classify each angle as acute, obtuse, right or straight



- a. Obtuse angle
 - b. Acute angle
 - c. Right angle
 - d. Straight angle
3. What is the measure of angle A in the right triangle below?



- a. 17
 - b. 27
 - c. 17
 - d. 90
4. What is the value of the side x in the right triangle below?



- a. 1
- b. 9
- c. 20
- d. 3

5. In a right triangle, the measure of one of the angles is 49° and the hypotenuse has a length of 50 cm. Which of the following is the nearest approximation to the length, in cm, of the leg opposite to this angle?
- 32.8
 - 57.5
 - 37.7
 - 30.3

Answer

1. (a) 2. (c) 3. (b) 4. (d) 5. (c)

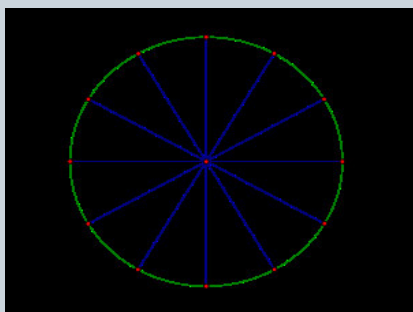
REVIEW QUESTIONS

- If A and B are the acute angles of right triangle ABC, show that $\sin A = \cos B$.
- Explain why the tangent of a 45° angle is 1.
- A Pythagorean triple is a set of three positive integers that make the equation $c^2 = a^2 + b^2$ true. Luz said that 3, 4, and 5 is a Pythagorean triple, and, for any positive integer k, 3k, 4k, and 5k is also a Pythagorean triple. Do you agree with Luz? Explain why or why not.
- What is the tangent ratio?
- Elaborate the cofunction theorem.

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RADIAN MEASURE AND CIRCULAR FUNCTIONS



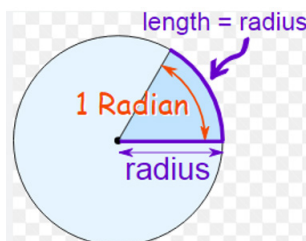
The radian as a common unit of angle measure in trigonometry. It is important that you become proficient converting back and forth between degrees and radians. Many units of measure come from seemingly arbitrary and archaic roots. Some even change over time. The meter, for example, was originally intended to be based on the circumference of the earth and now has an amazingly complicated scientific definition. We typically use degrees to measure angles. A degree is $1/360$ th of a complete rotation around a circle. Radians are alternate units used to measure angles in trigonometry.

After studying this chapter, you will understand the core concepts of:

- Radian Measure
- Circular Functions

INTRODUCTION

In measuring angles, we generally use two types of measuring units, namely degrees and radians. Both measurements are widely used units in geometry and trigonometry in solving many problems. In general, the degree measure is used in earlier classes of mathematics since it is easy to understand and remember as this involves use of a protractor to measure the angle. In case of radians, one should understand the concept of circles and related terms along with their meaning and formulas.



As we know, equal arcs of a circle subtend equal angles at the center. Considering a circle.

Radius = r

Arc length = l

Angle = 1 radian

From this, we can say that an arc of length l will subtend an angle whose measure is l/r radian.

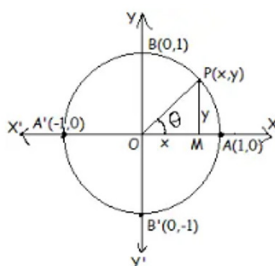
Let r be the radius of a circle, l be its arc length subtends an angle θ radian at the center. Thus,

$$\theta = l/r$$

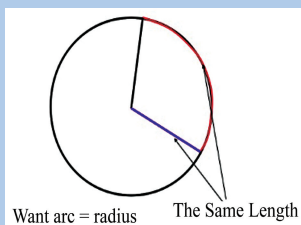
Or

$$l = r \theta$$

Circular functions are the functions whose domain are the set of numbers that correspond to the measures of angles of with respect to the trigonometric functions. The ranges of these circular functions are the set of numbers that are real numbers with respect to the trigonometric functions. Trigonometric functions are called circular-functions because the radian measures of the angles are calculated by the arc length of the circle.



The unit circle $x^2 + y^2 = 1$ shown in the above diagram. Let A be the point on the X-axis where circle touches the X-axis. So the coordinate of point A will be (1,0). Let P be any point on the circle whose coordinates be (x,y). The distance of point P is greater than zero ($p > 0$) for the counter clockwise and in clockwise direction $P < 0$. The x-coordinate of the point P will be define as cosine and Y coordinate of the point P will be sine.



CORE CONCEPT: RADIAN MEASURE

Radian measure is the ratio of the length of a circular arc to the radius of the arc. An angle with its vertex at the center of a circle that intercepts an arc on the circle equal in length to the radius of the circle has a measure of 1 radian.

A radian is the angle subtended at the center of a circle by an arc of the circle equal in length to its radius. It is proved in geometry that equal central angles of two circles subtend arcs proportional to their radii; and the converse is true. Hence the radian is independent of the length of the radius. The illustration represents two circles of radius r . Arc AB of length r subtends 1 radian (rad) at the center O of the circle, and arc $A'B'$ of length s subtends θ rad at its center. Since arcs on equal circles are proportional to their subtended central angles, $s/r = \theta/1$, or below formula

$$S = r\theta$$

holds. If $\theta = 2\pi$, $s = 2\pi r$, the circumference of the circle. Therefore 2π rad is the complete angle about a point or 360° , and

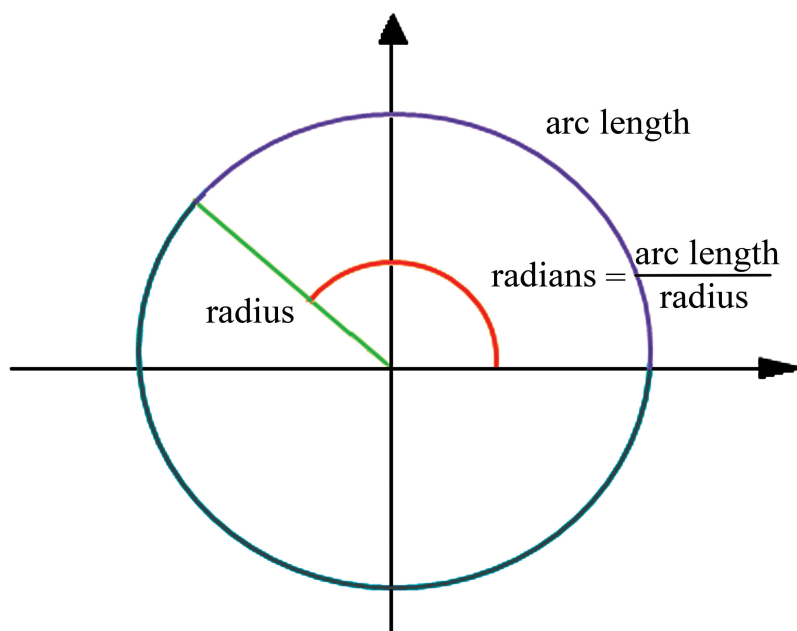
$$\begin{aligned} 2\pi \text{ rad} &= 360^\circ \\ 1 \text{ rad} &= 360^\circ / 2\pi = 57.2958^\circ = 57^\circ 17' 45'' \\ 1^\circ &= 2\pi / 360 \text{ rad} = 0.0174533 \text{ rad} \end{aligned}$$

Observe that

$$\begin{aligned} 30^\circ &= 30\pi / 180 \text{ rad} = \pi / 6 \text{ rad} \\ 45^\circ &= \pi / 4 \text{ rad} \\ 60^\circ &= \pi / 3 \text{ rad} \\ 90^\circ &= \pi / 2 \text{ rad} \\ 135^\circ &= 3\pi / 4 \text{ rad} \end{aligned}$$

You can enter radians directly into your calculator to evaluate a trigonometric function at an angle in radians, but you must make sure your calculator is in radian mode. Students should be reminded to check what mode their calculator is in when they are doing problems involving the trigonometric functions.

Radian measure is an alternative way to measure angles. Instead of dividing the circle into an arbitrary number of parts, we look at the length of the arc that subtends the angle. We measure the angle based on this length as a ratio to the radius.



One radian is defined as the angle where the length of the arc equals the length of the radius. If we traverse the circle completely, we will have travelled the length of the circumference.

$$C = 2\pi r$$

Using this, we can find the number of radians in a circle. The length of the circumference, as a ratio to the radius, is:

$$\text{Radians in a circle} = \text{arc length} / \text{radius}$$

Applying this to a circle, if we rearrange this equation to isolate radians, we can find the number of radians in a given number of degrees:

$$\text{Radians} = 360 \text{ degrees} / 2\pi$$

$$= 180 / \pi \text{ degrees}$$

Rearrange to obtain:

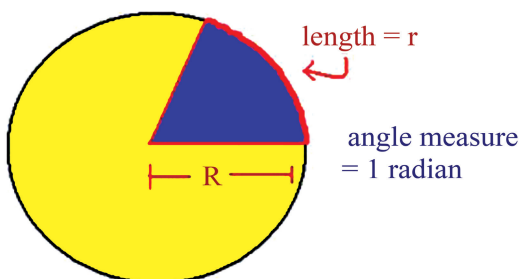
$$\text{Degrees} = \pi / 180 \text{ radians}$$

Using this formula, we can find the number of degrees in a given number of radians. To summarize, radian measure is an alternative way to measure angles. Instead of dividing the circle into an arbitrary number of parts, we look at the length of the arc that subtends the angle. We measure the angle based on this length as a ratio to the radius. We can convert between radian and degree measure using the relationship that 2π radians equals 360 degrees; they both measure the angles in a circle:

$$2\pi \text{ radians} = 360 \text{ degrees}$$

Definition of Radian

Radians are the standard mathematical way to measure angles. One radian is equal to the angle created by taking the radius of a circle and stretching it along the edge of the circle.



The radian is a pure mathematical measurement and, therefore, is preferred by mathematicians over degree measures. For use in everyday work, the degree is easier to work with, but for purely mathematical pursuits, the radian gives better results. You probably will never see radian measures used in construction or surveying, but it is a common unit in mathematics and physics. The concept of measuring an angle by the length of the arc was first made popular in the early 1700s by Roger Cotes, an English mathematician who worked closely with Isaac Newton, but the term radian was not used to describe this concept until the late 1800s by James Thomson at Queen's College, Belfast, Ireland.

Students should have a deal of practice in finding the trigonometric functions of angles expressed in radians. Since students are more familiar with degrees, it is often best to convert back to degrees.

Convert Degrees to Radians

The unit used to describe the measurement of an angle that is most familiar is the degree. To convert radians to degrees or degrees to radians, the following relationship can be used:

$$\text{Angle in degrees} = \text{angle in radians} * (180 / \pi)$$

This equation works because a half circle is 180 degrees and is also equal to π radians.

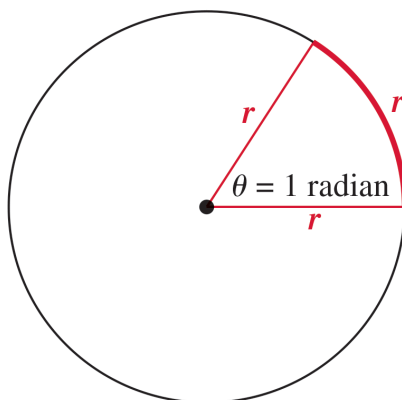
$$180 \text{ degrees} = \pi \text{ radians}$$

$$1 \text{ radian} = 180 / \pi$$

The Radian Measure of an Angle

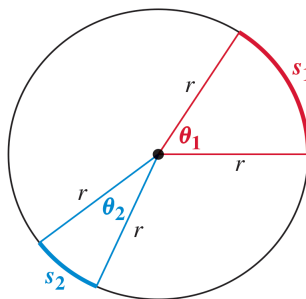
In geometry and most everyday applications, angles are measured in degrees. However, radian measure is another way to measure angles. Using radian measure allows us to write trigonometric functions as functions not only of angles but also

of real numbers in general. We defined one full rotation as an angle having measure 360° . Now we think of the angle in the context of a circle. A central angle is an angle that has its vertex at the center of a circle.



When the intercepted arc's length is equal to the radius, the measure of the central angle is 1 radian. From geometry, we know that the ratio of the measures of two angles is equal to the ratio of the lengths of the arcs subtended by those angles (along the same circle).

$$\frac{\theta_1}{\theta_2} = \frac{s_1}{s_2}$$



If $\theta_1 = 1$ radian, then the length of the subtended arc is equal to the radius, $s_1 = r$. This leads to a general definition of radian measure. Note that both s and r are measured in units of length. When both are given in the same units, the units cancel, giving the number of radians as a dimensionless (unit less) real number. One full rotation corresponds to an arc length equal to the circumference $2\pi r$ of the circle with radius r . We see then that one full rotation is equal to 2π radians.

$$\theta_{\text{full rotation}} = \frac{2\pi r}{r} = 2\pi$$

Because radians are unit less, the word radians (or rad) is often omitted. If an angle measure is given simply as a real number, then radians are implied.

WORDS

The measure of θ is 4 degrees.

The measure of θ is 4 radians.

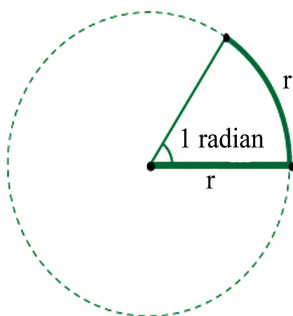
MATH

$$\theta = 4^\circ$$

$$\theta = 4$$

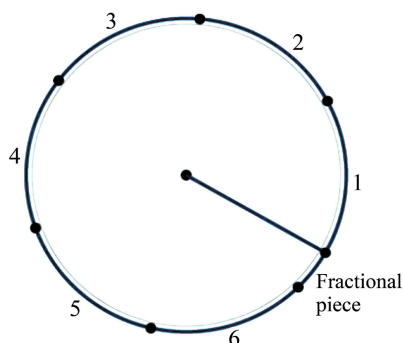
Converting Between Degrees and Radians

To convert between degrees and radians, we must first look for a relationship between them. We start by considering one full rotation around the circle. An angle corresponding to one full rotation is said to have measure 360° , and we saw previously that one full rotation corresponds to $\theta = 2\pi$ rad.



Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An arc may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the circumference of that **circle**.

The circumference of a circle is $C=2\pi r$. If we divide both sides of this equation by r , we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 1.



Circle is a simple closed shape. It is the set of all points in a plane that are at a given distance from a given point, the center; equivalently it is the curve traced out by a point that moves so that its distance from a given point is constant.

Figure 1

This brings us to our new angle measure. One radian is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = \frac{360^\circ}{2} = 180^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ$$

Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

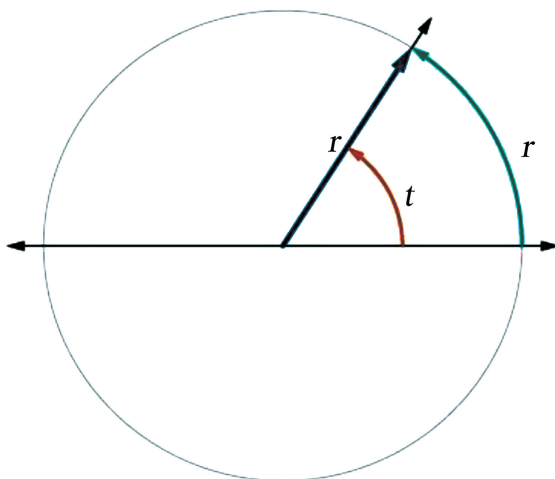


Figure 2: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An arc length s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius. This ratio, called the radian measure, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s to the radius r .

One radian is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle.

$$s = r\theta$$

$$\theta = \frac{s}{r}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

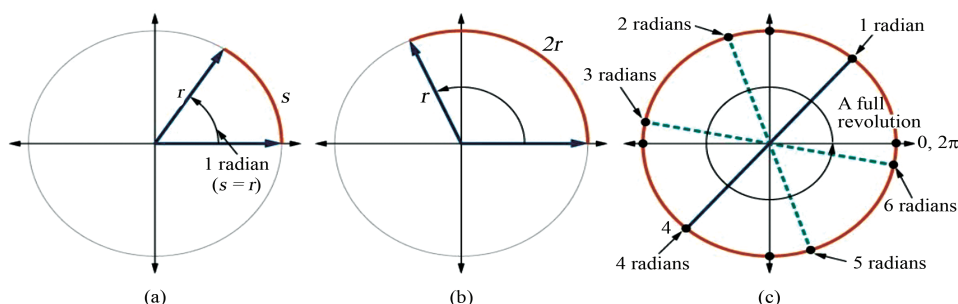


Figure 3: (a) In an angle of 1 radian, the arc length s equals the radius r . (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 4.

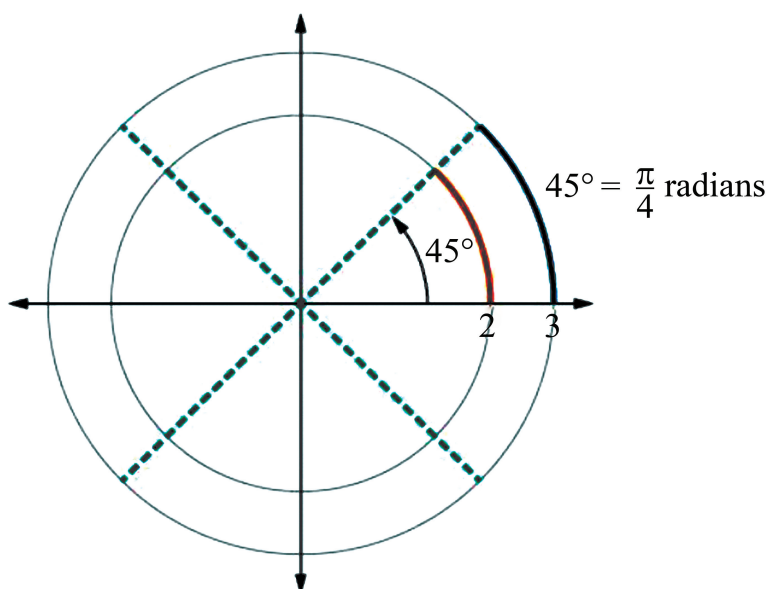


Figure 4: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\begin{aligned} \text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} &= \frac{1}{4}\pi \\ \text{Larger circle: } \frac{\frac{3}{4}\pi}{3} &= \frac{1}{4}\pi \end{aligned}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

Using Radians

Because radian measure is the ratio of two lengths, it is a unit less measure. For example, in Figure 3, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the “inches” cancel, and we have a result without units. Therefore, it is not necessary to write the label “radians” after a radian measure, and if we see an angle that is not labeled with “degrees” or the degree symbol, we can assume that it is a radian measure. Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360° . We can also track one rotation around a circle by finding the circumference, $C=2\pi r$, and for the unit circle $C=2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$1 \text{ rotation} = 360^\circ = 2\pi \text{ radians}$
 $\frac{1}{2} \text{ rotation} = 180^\circ = \pi \text{ radians}$
 $\frac{1}{4} \text{ rotation} = 90^\circ = \frac{\pi}{2} \text{ radians}$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 5. Memorizing these angles will be very useful as we study the properties associated with angles.

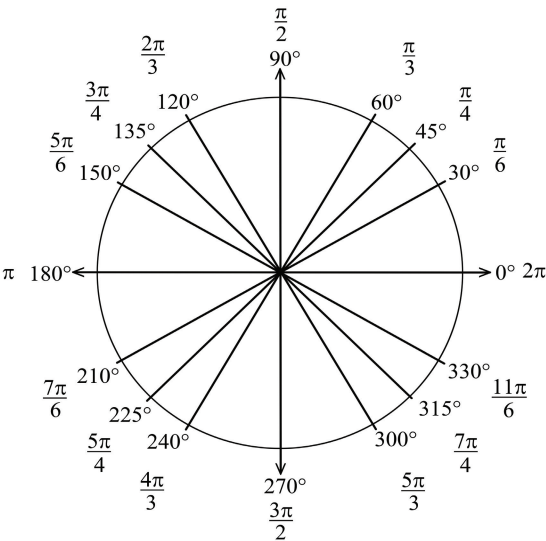


Figure 5: Commonly encountered angles measured in degrees.

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

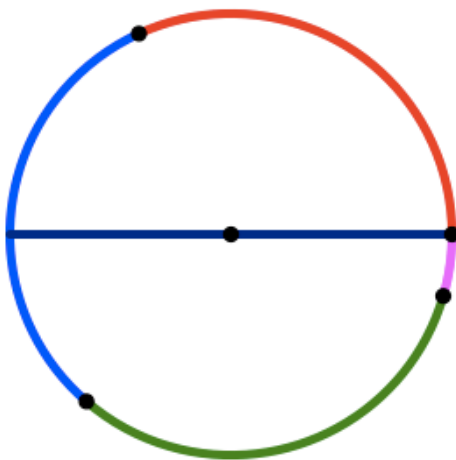
$$\frac{\theta}{180} = \frac{\theta^R}{\pi}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π . Or, phrased another way, degrees is to 180 as radians is to π .

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi}$$

Finding Conterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360° , or 0 to 2π . It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.



It is possible for more than one angle to have the same terminal side. Look at Figure 6. The angle of 140° is a positive angle, measured counterclockwise. The angle of -220° is a negative angle, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are conterminal angles. Every angle greater than 360° or less than 0° is conterminal with an angle between 0° and 360° , and it is often more convenient to find the conterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

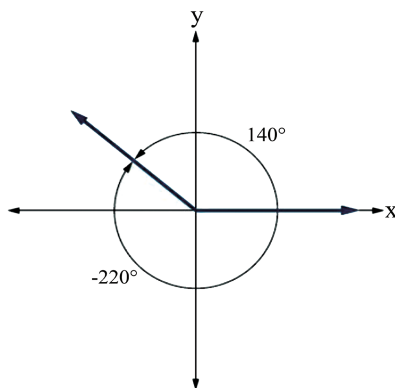


Figure 6: An angle of 140° and an angle of -220° are coterminal angles.

Any angle has infinitely many coterminal angles because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is -260° . Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions. An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 7 for examples of reference angles for angles in different quadrants.

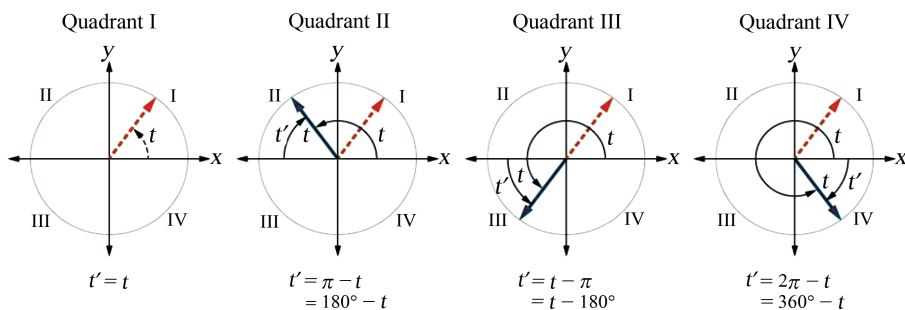
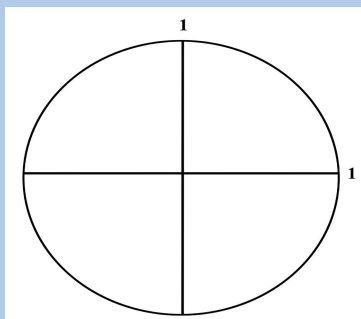


Figure 7

Finding Coterminal Angles Measured in Radians

We can find coterminal angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations. Given an angle greater than 2π , find a coterminal angle between 0 and 2π .

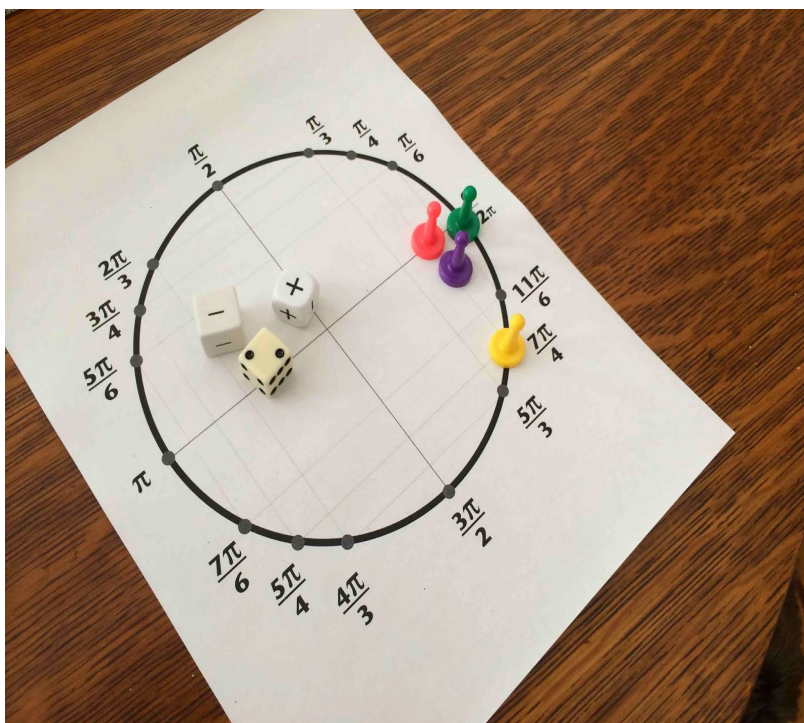
- Subtract 2π from the given angle.
- If the result is still greater than 2π , subtract 2π again until the result is between 0 and 2π .
- The resulting angle is coterminal with the original angle.



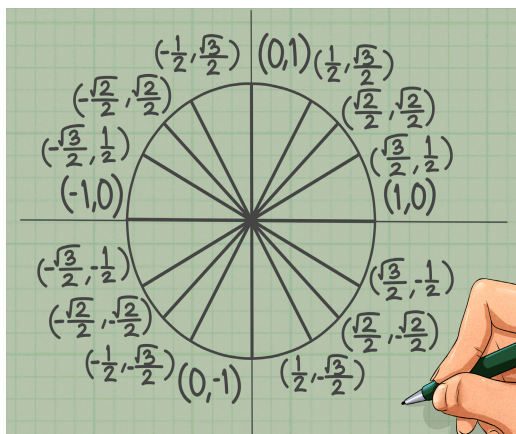
CORE CONCEPT: CIRCULAR FUNCTIONS

The trigonometric functions (also called circular functions, angle functions or goniometric functions) are functions of an angle. They relate the angles of a triangle to the lengths of its sides. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.

The graph of the equation $x^2 + y^2 = 1$ is a circle in the rectangular coordinate system. This graph is called the unit circle and has its center at the origin and has a radius of 1 unit. Trigonometric functions are defined so that their *domains are sets of angles* and their ranges are sets of real numbers. Circular functions are defined such that their *domains are sets of numbers* that correspond to the measures (in radian units) of the angles of analogous trigonometric functions. The ranges of these circular functions, like their analogous trigonometric functions, are sets of real numbers. These functions are called circular functions because radian measures of angles are determined by the lengths of arcs of circles. In particular, trigonometric functions defined using the unit circle lead directly to these circular functions.

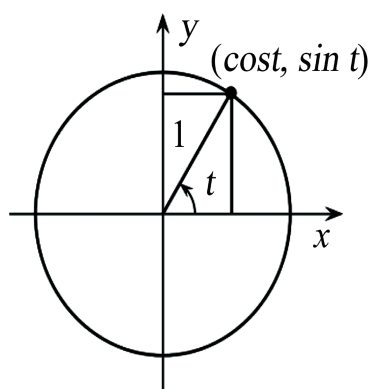


The circle below is drawn in a coordinate system where the circle's center is at the origin and has a radius of 1. This circle is known as a unit circle.

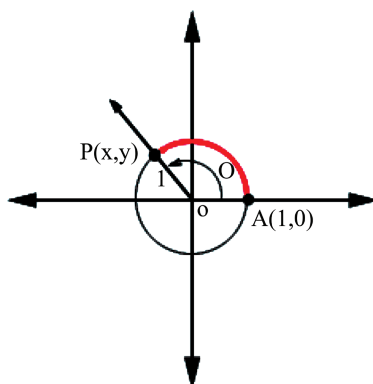


The x and y coordinates for each point along the circle may be ascertained by reading off the values on the x and y axes. If you picture a **right triangle** with one side along the x-axis:

Right triangle is a triangle in which one angle is a right angle. The relation between the sides and angles of a right triangle is the basis for trigonometry.



Then the cosine of the angle would be the x-coordinate and the sine of the angle would be the y-coordinate. Since both the coordinates are defined by using a unit circle, they are often called circular functions. We can apply the six trigonometric functions to circles as well. If you are using a calculator, you will have to change the mode to radians.



Consider the circle $x^2 + y^2 = 1$. Let A be the point (1, 0). Given any real number s , start at A and measure s to a point P (x , y) in a counterclockwise direction if s is positive or clockwise if s is negative. s is the arc length. The sine and cosine of s are defined by the coordinates of point P. The tangent, secant, cotangent, and cosecant are then defined in terms of sine and cosine:

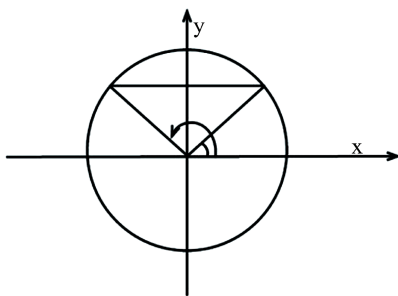
- $\sin s = y$
- $\cos s = x$
- $\tan s = \sin s / \cos s$ (if $\cos s \neq 0$)
- $\cot s = \cos s / \sin s$ (if $\sin s \neq 0$)
- $\sec s = 1 / \cos s$ (if $\cos s \neq 0$)
- $\csc s = 1 / \sin s$ (if $\sin s \neq 0$)

You probably noticed that the functions refer to s and not the angle Θ . The circular functions of s are equal to the same functions of the angle Θ , measured in radians, provided that the radius of the circle is one. For the purposes of this course, the radius of the circle will always be one unless mentioned otherwise.

Example

Solve the equation $\sin v = 0.5$ with the unit circle.

If we examine the figure below, it is evident that there are two solutions to the problem:



We arrive at the first solution by using a pocket calculator and keying:

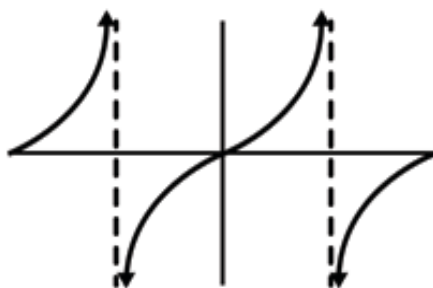
$$v = \sin^{-1} 0.5 = 30^\circ \quad v = \sin^{-1} 0.5 = 30^\circ$$

Since half a revolution is 180 degrees, we ascertain the other angle by:

$$180 - v = 180 - 30 = 150^\circ$$

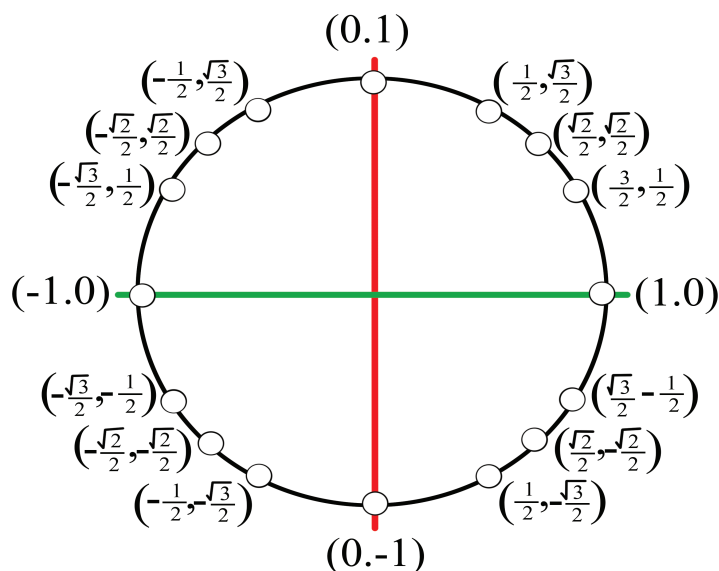
Properties of Circular Functions

The unit circle has as its center the origin point of the Cartesian coordinates x and y , and has a radius 1. The trigonometric functions are also called circular functions, because they describe relationships between angles on the unit circle. We use different ways to describe trigonometry in order to see how the relationships and equations apply to different situations.



Special Values of the Trigonometric Functions

When the trigonometric functions are defined as circular, the special values of trigonometric functions can be seen on the unit circle. If t , the length of the arc from the coordinates $(1, 0)$ equals 0, then $\sin t$ equals 0, $\cos t$ equals 1, and tangent t equals 0. This is because $\sin t$ equals y , $\cos t$ equals x , and $\tan t$ equals y/x . When t equals 0, y/x equals $0/1$. If t equals $\pi/4$, then its coordinates are $([\sqrt{2}/2], [\sqrt{2}/2])$. Therefore, $\sin t$ is $(\sqrt{2})/2$, $\cos t$ is $(\sqrt{2})/2$, and tangent t is 1.



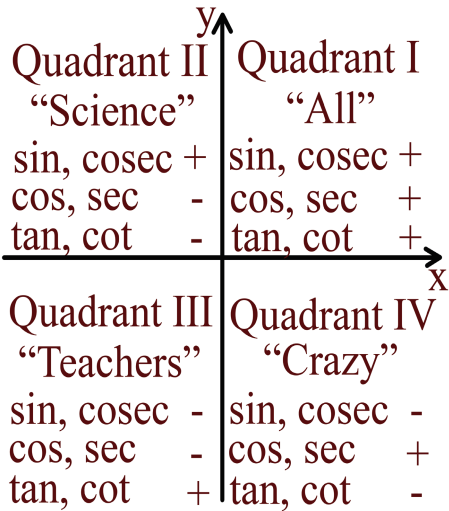
Domains of the Trigonometric Functions

If a circular function results in division by zero, it is undefined. Therefore, each function has a specific domain. Sines and cosines are defined for all real values of t . Tangents and secants are defined for all real numbers, except when the value of x is 0. This happens when the value of t equals $(\pi/2) + n\pi$. Similarly, cosecants (csc) and cotangents (cot), are defined for all real numbers other than $n\pi$, which happens when x equals 0.

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
$\cot \theta$	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
$\operatorname{cosec} \theta$	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

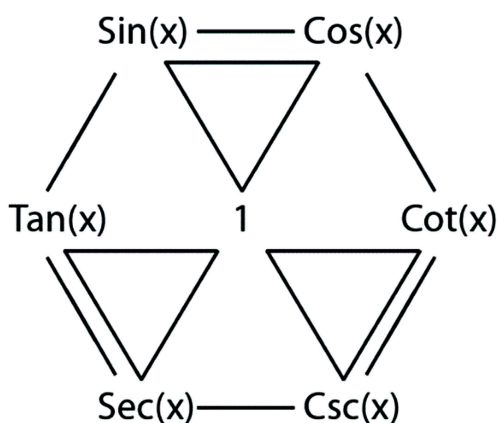
Sign of the Trigonometric Functions

All other values of the trigonometric functions can also be seen on the unit circle. The unit circle is divided into quadrants by the Cartesian coordinates, so the signs of each circular function can be determined by the value of t. If the value of t has a positive value for both x and y, then it lies in Quadrant I. In Quadrant II, only the value of y is positive, and x has a negative value, so sin t and cosecant t will be positive. All other functions will be negative. In Quadrant III, both coordinates for x and y are negative. This means that tangents and their reciprocal cotangents will be positive. In Quadrant 4, x is positive and y is negative, so cosines and their reciprocal, secants, will be positive. There are several ways to remember the direction of the signs.



Fundamental Identities

The trigonometric functions are related to one another because the formula for the unit circle is $x^2 + y^2 = 1$. Therefore, the cosecant of t (csc t) is defined as $1/\sin t$; the secant of t (sec t) as $1/\cos t$, and the cotangent of t (cot t) as $1/\tan t$. Similarly, $\sin^2 t + \cos^2 t = 1$, $\tan^2 t + 1 = \sec^2 t$, and $1 + \cot^2 t = \csc^2 t$.

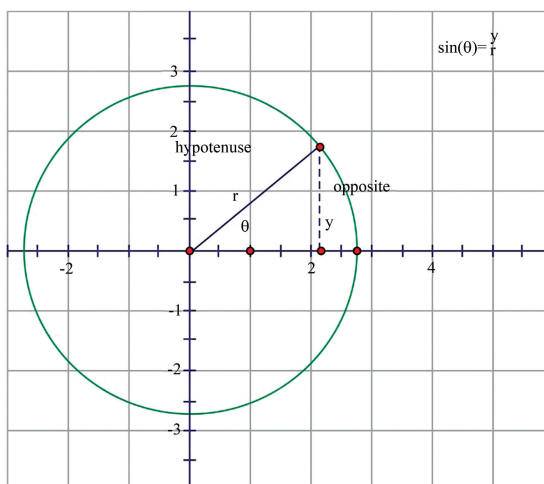


The Sine Graph

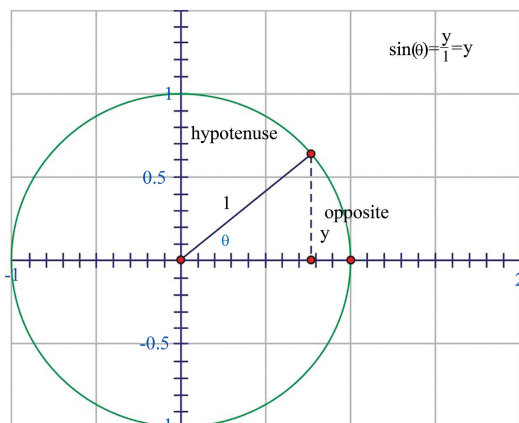
By now, you have become very familiar with the specific values of sine, cosine, and tangents for certain angles of rotation around the coordinate grid. In mathematics, we can often see a lot by looking at how one quantity changes as we consistently vary another. We will be looking at the sine value as a function of the angle of rotation around the coordinate plane. We refer to any such function as a circular function, because they can be defined using the **unit circle**. Recall from earlier sections

Unit circle is a circle with a radius of one. Frequently, especially in trigonometry, the unit circle is the circle of radius one centered at the origin $(0, 0)$ in the Cartesian coordinate system in the Euclidean plane.

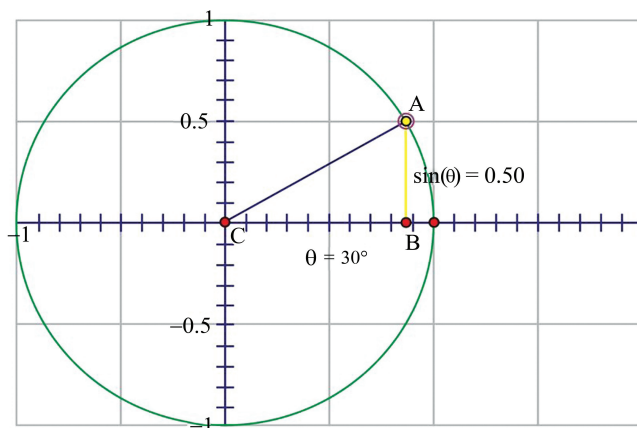
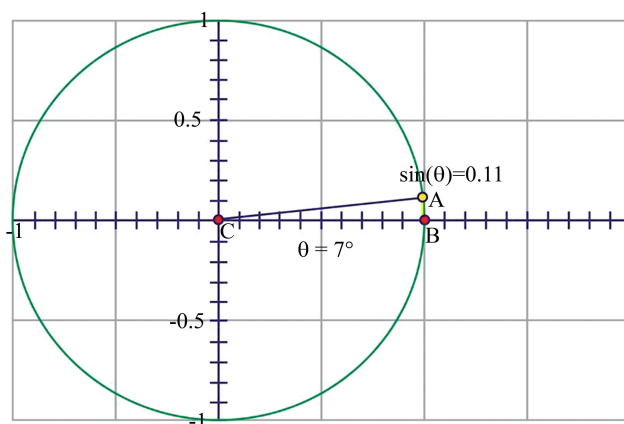
that the sine of an angle in standard position is the ratio of $\frac{y}{r}$, where y is the y -coordinate of any point on the circle and r is the distance from the origin to that point.

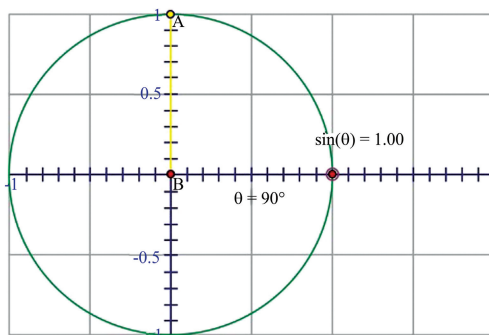
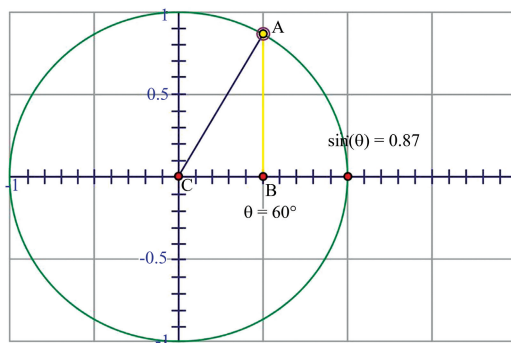


Because the ratios are the same for a given angle, regardless of the length of the radius r , we can use the unit circle as a basis for all calculations.

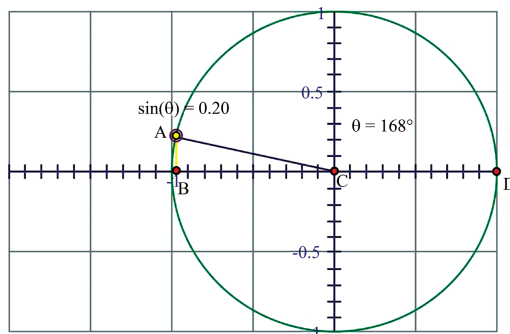
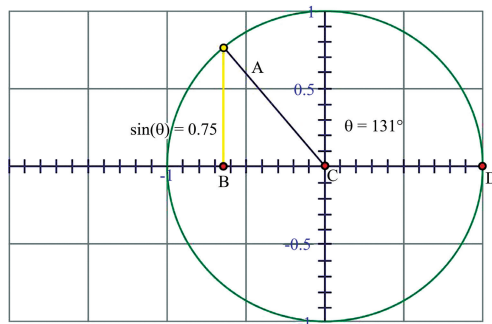


The denominator is now 1, so we have the simpler expression, $\sin \theta = y$. The advantage to this is that we can use the y -coordinate of the point on the unit circle to trace the value of $\sin \theta$ through a complete rotation. Imagine if we start at 0 and then rotate counter-clockwise through gradually increasing angles. Since the y -coordinate is the sine value, watch the height of the point as you rotate. Through Quadrant I that height gets larger, starting at 0, increasing quickly at first, then slower until the angle reaches 90° , at which point, the height is at its maximum value, 1.

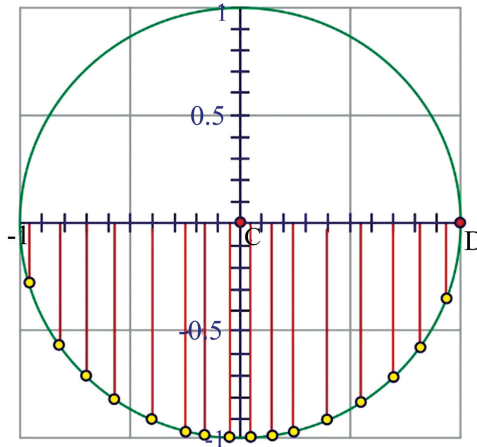




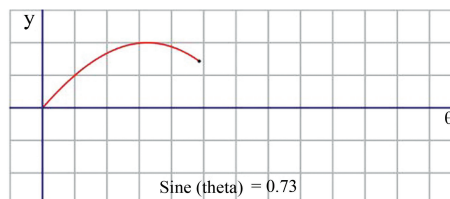
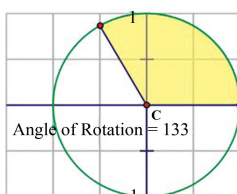
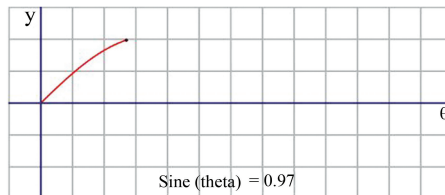
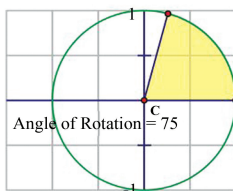
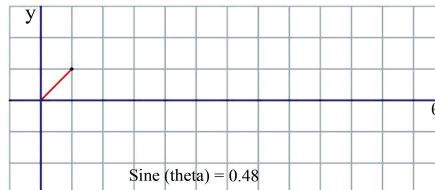
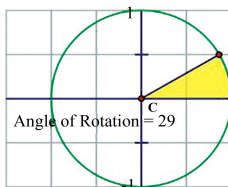
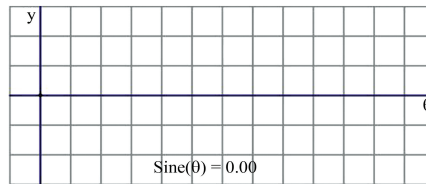
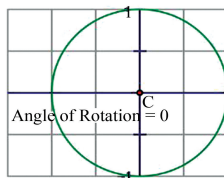
As you rotate into the second quadrant, the height starts to decrease towards zero.



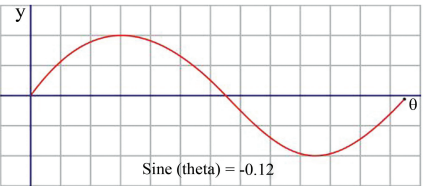
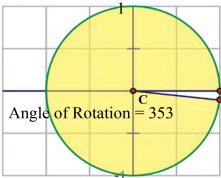
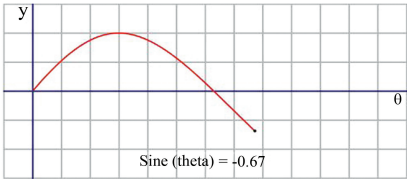
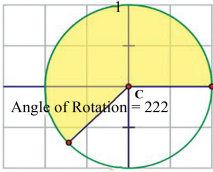
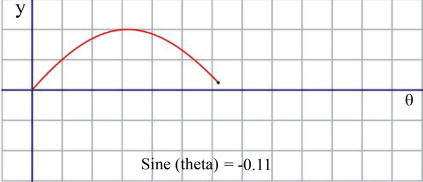
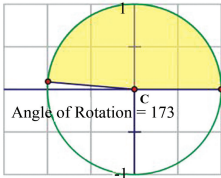
When you start to rotate into the third and fourth quadrants, the length of the segment increases, but this time in a negative direction, growing to -1 at 270° and heading back toward 0 at 360° .



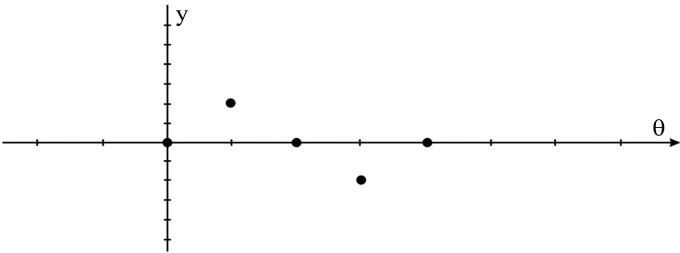
After one complete rotation, even though the angle continues to increase, the sine values will repeat themselves. The same would have been true if we chose to rotate clockwise to investigate negative angles, and this is why the sine function is a periodic function. The period is 2π because that is the angle measure before the sine of the angle will repeat its values. Let's translate this circular motion into a graph of the sine value vs. the angle of rotation. The following sequence of pictures demonstrates the connection. These pictures plot $(\theta, \sin \theta)$ on the coordinate plane as (x, y) .



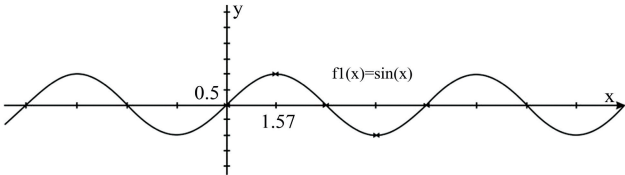
Rotation is a circular movement of an object around a center (or point) of rotation.



After we rotate around the circle once, the values start repeating. Therefore, the sine curve, or “wave,” also continues to repeat. The easiest way to sketch a sine curve is to plot the points for the quadrant angles. The value of $\sin \theta$ goes from 0 to 1 to 0 to -1 and back to 0. Graphed along a horizontal axis, it would look like this:



Filling in the gaps in between and allowing for multiple rotations as well as negative angles results in the graph of $y = \sin x$ where x is any angle of **rotation**, in radians.



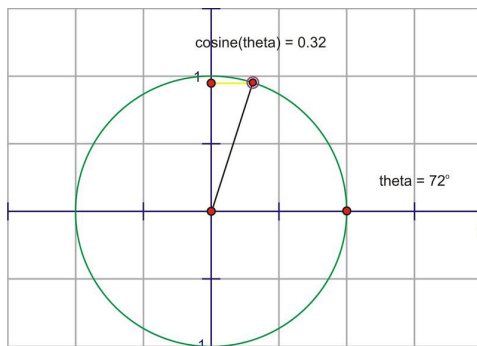
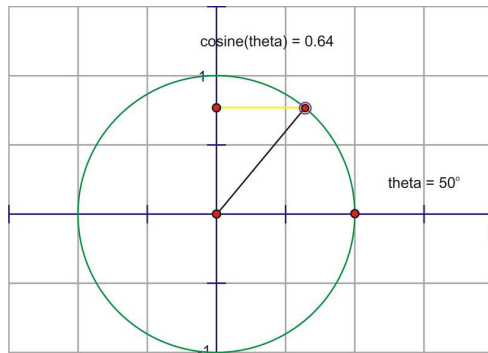
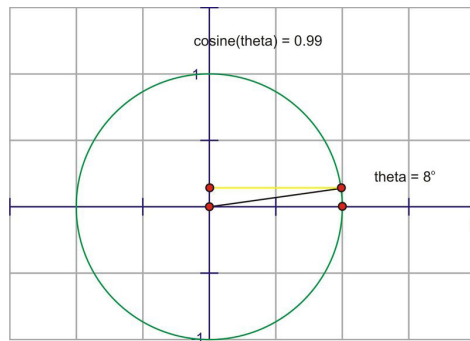
As we have already mentioned, $\sin x$ has a period of 2π . You should also note that the y -values never go above 1 or below -1, so the range of a sine curve is $\{-1 \leq y \leq 1\}$. Because angles can be any value and will continue to rotate around the

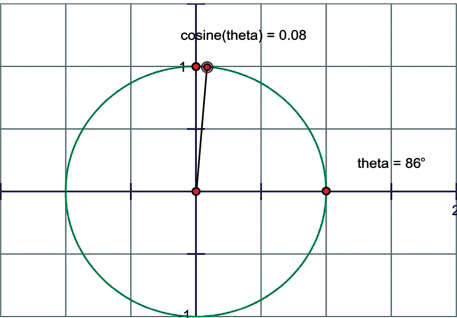
circle infinitely, there is no restriction on the angle x , so the domain of $\sin x$ is all reals.

The Cosine Graph

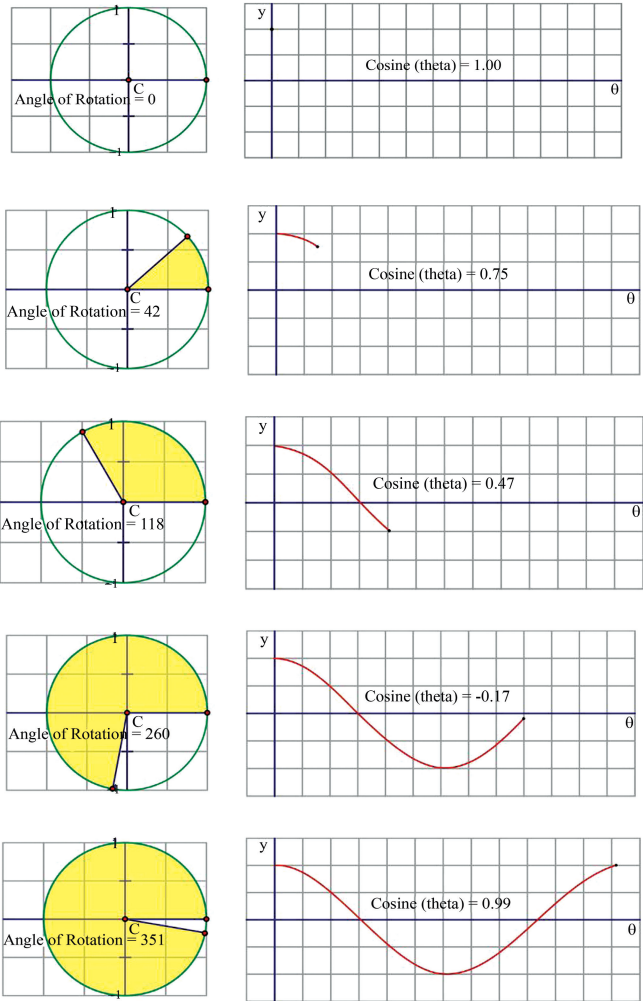
The cosine of an angle is the same as the sine of its complementary angle. So, it should not be a surprise that sine and cosine waves are very similar in that they are both periodic with a period of 2π , a range from -1 to 1 , and a domain of all real

angles. The cosine of an angle is the ratio of $\frac{x}{r}$, so in the unit circle, the cosine is the x -coordinate of the point of rotation. If we trace the x -coordinate through a rotation, notice the change in the distance is similar to $\sin x$, but $\cos x$ starts at one instead of zero. The x -coordinate at 0° is 1 and the x -coordinate for 90° is 0 , so the cosine value is decreasing from 1 to 0 through the 1st quadrant.

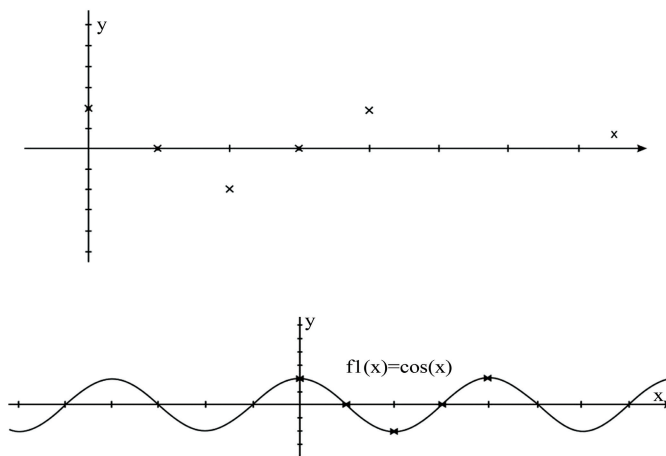




Here is a similar sequence of rotations to the one used for sine. This time compare the x– coordinate of the point of rotation with the height of the point as it traces along the horizontal. These pictures plot $(\theta, \cos \theta)$ on the coordinate plane as (x, y) .



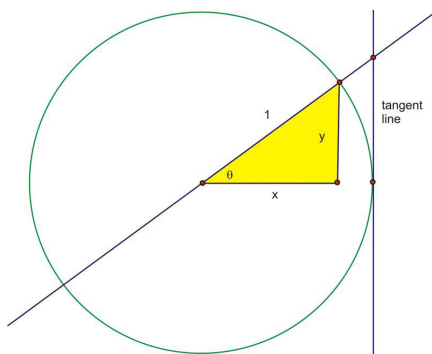
Plotting the quadrant angles and filling in the in-between values shows the graph of $y=\cos x$



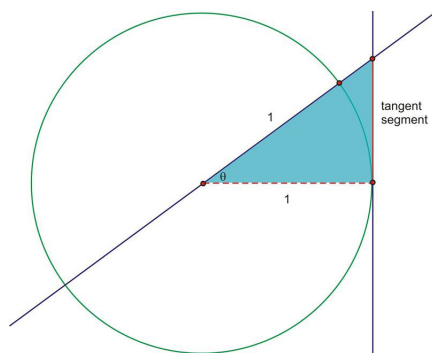
The graph of $y = \cos x$ has a period of 2π . Just like $\sin x$, the range of a cosine curve is $\{-1 \leq y \leq 1\}$ and the domain of $\cos x$ is all reals. Notice that the shape of the curve is exactly the same, but *shifted* by $\frac{\pi}{2}$.

The Tangent Graph

The name of the tangent function comes from the tangent line of a circle. This is a line that is perpendicular to the radius at a point on the circle so that the line touches the circle at exactly one point.

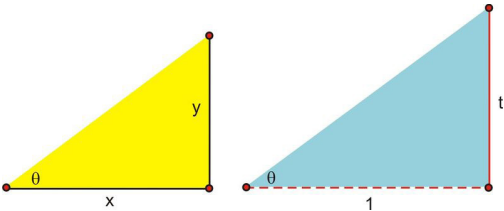


If we extend angle θ through the unit circle so that it intersects with the tangent line, the tangent function is defined as the length of the red segment.



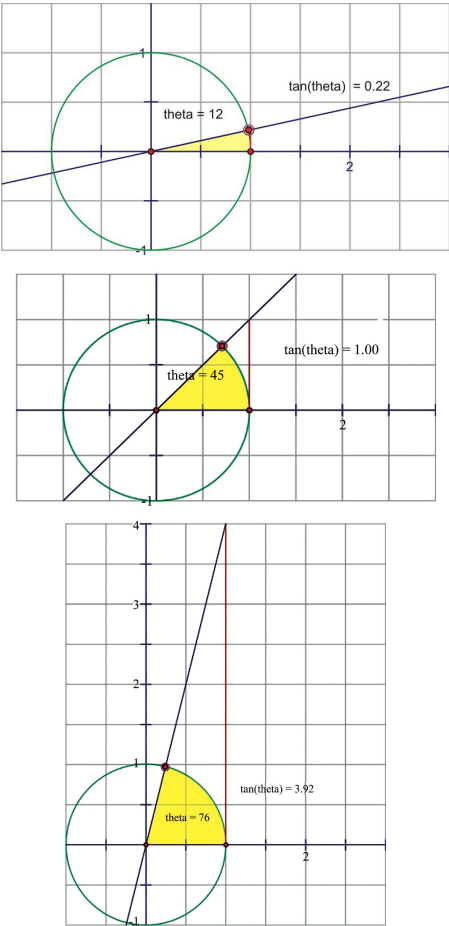
The dashed segment is 1 because it is the radius of the unit circle. Recall that

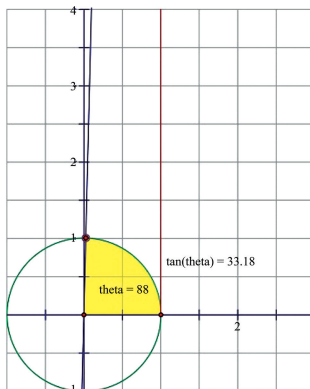
the $\tan \theta = \frac{y}{x}$, and it can be verified that this segment is the tangent by using similar triangles.



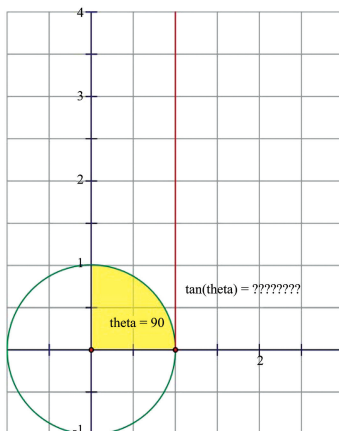
$$\tan \theta = \frac{y}{x} = \frac{t}{1} = t$$
$$\tan \theta = t$$

So, as we increase the angle of rotation, think about how this segment changes. When the angle is zero, the segment has no length. As we rotate through the first quadrant, it will increase very slowly at first and then quickly get very close to one, but never actually touch it.

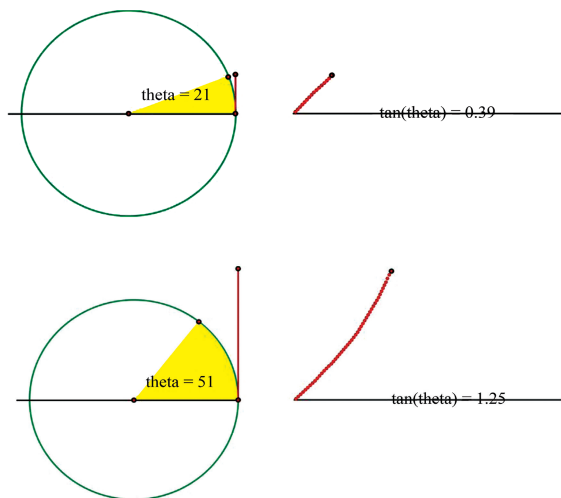


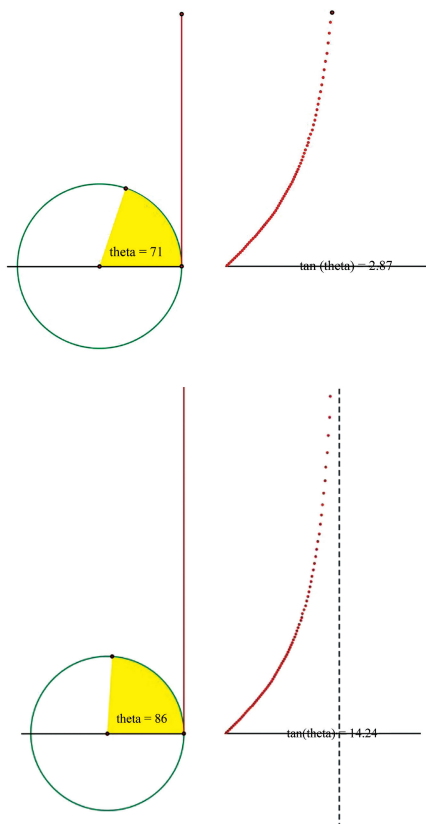


As we get very close to the y-axis the segment gets infinitely large, until when the angle really hits 90° , at which point the extension of the angle and the tangent line will actually be parallel and therefore never intersect.



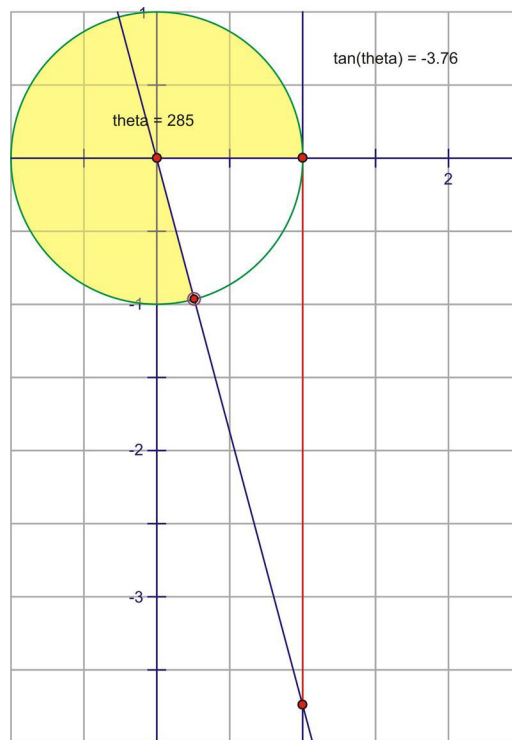
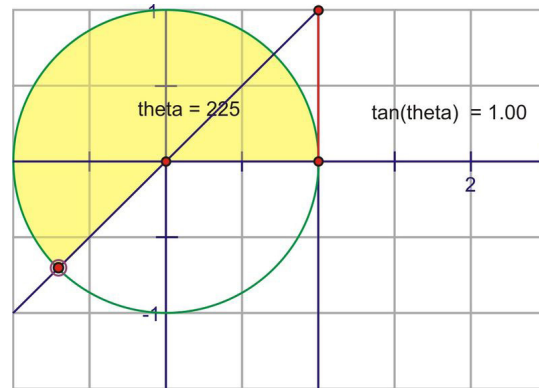
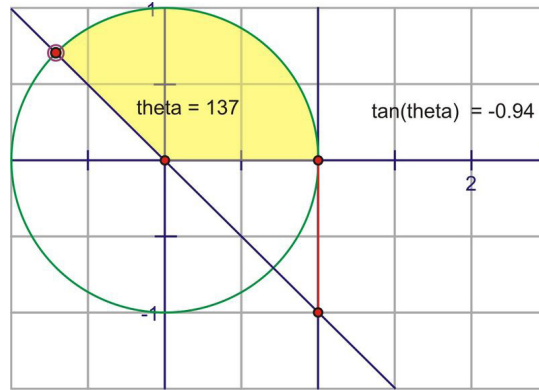
This means there is no finite length of the tangent segment, or the tangent segment is infinitely large. Let's translate this portion of the graph onto the coordinate plane. Plot $(\theta, \tan \theta)$ as (x, y) .



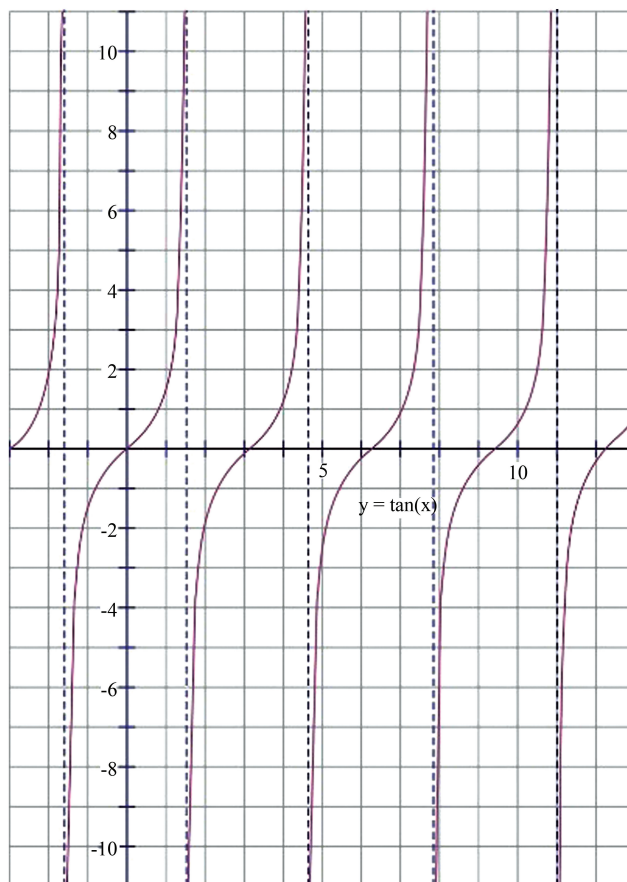


Quadrant is a circular sector equal to one quarter of a circle, or half a semicircle. Quadrant (plane geometry), a sector of a two-dimensional Cartesian coordinate system.

In fact as we get infinitely close to 90° , the tangent value increases without bound, until when we actually reach 90° , at which point the tangent is undefined. Recall there are some angles (90° and 270° , for example) for which the tangent is not defined. Therefore, at these points, there are going to be vertical asymptotes. Rotating past 90° , the intersection of the extension of the angle and the tangent line is actually below the x-axis. This fits nicely with what we know about the tangent for a 2nd quadrant angle being negative. At first, it will have very large negative values, but as the angle rotates, the segment gets shorter, reaches 0, then crosses back into the positive numbers as the angle enters the 3rd quadrant. The segment will again get infinitely large as it approaches 270° . After being undefined at 270° , the angle crosses into the 4th **quadrant** and once again changes from being infinitely negative, to approaching zero as we complete a full rotation.



The graph $y = \tan x$ over several rotations would look like this:



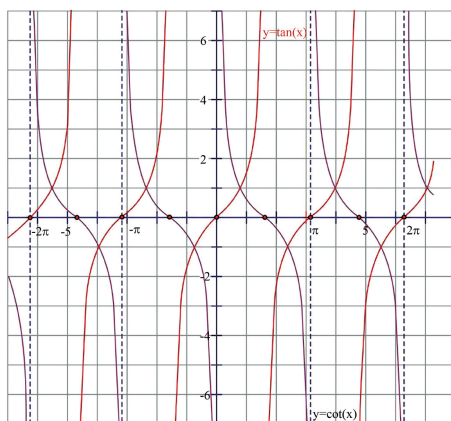
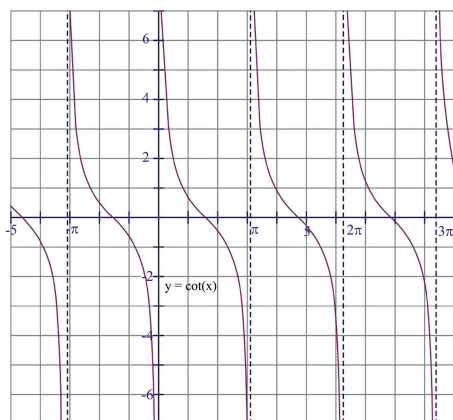
Notice the x -axis is measured in radians. Our asymptotes occur every π radians, starting at $\frac{\pi}{2}$. The period of the graph is therefore π radians. The domain is all reals except for the asymptotes at $\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}$, etc. and the range is all real numbers.

The Three Reciprocal Functions

For the three reciprocal functions, it gets increasingly difficult to show the segment representation on the unit circle. Instead of going through all of this, we will show the $\cot x$, $\csc x$, and $\sec x$ through the graphs of their reciprocal functions, $\tan x$, $\sin x$, and $\cos x$.

Cotangent

Cotangent is the reciprocal of tangent, $\frac{x}{y}$, so it would make sense that where ever the tangent had an asymptote, now the cotangent will be zero. The opposite of this is also true. When the tangent is zero, now the cotangent will have an asymptote. The shape of the curve is generally the same, so the graph looks like this:

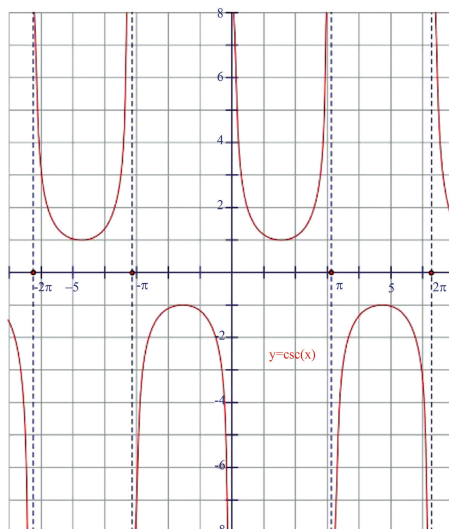


Sine function is one of the basic functions encountered in trigonometry (the others being the cosecant, cosine, cotangent, secant, and tangent).

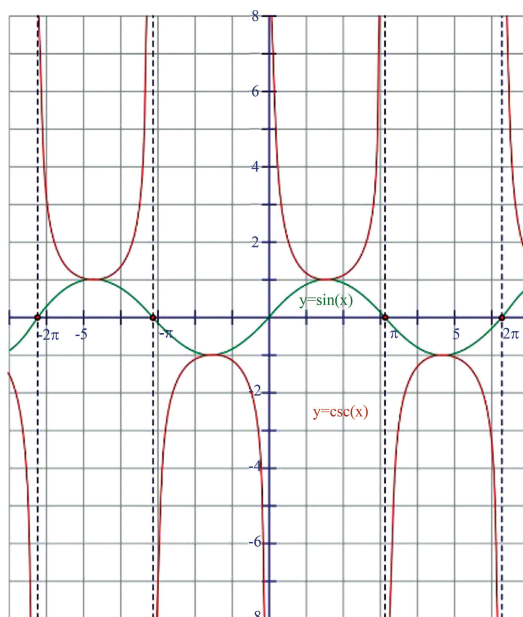
When you overlap the two functions, notice that the graphs consistently intersect at 1 and -1. These are the angles that have 45° as reference angles, which always have tangents and cotangents equal to 1 or -1. It makes sense that 1 and -1 are the only values for which a function and its reciprocal are the same. Keep this in mind as we look at cosecant and secant compared to their reciprocals of sine and cosine. The cotangent function has a domain of all real angles except multiples of $\pi\{\dots-2\pi, -\pi, 0, \pi, 2\pi, \dots\}$. The range is all real numbers.

Cosecant

Cosecant is the reciprocal of sine, or $\frac{1}{y}$. Therefore, whenever the sine is zero, the cosecant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the **sine function** in the same quadrants. Here is the graph.



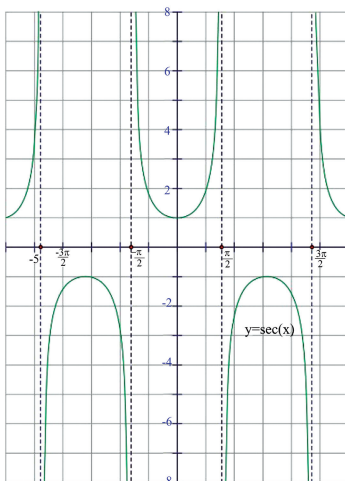
The period of the function is 2π , just like sine. The domain of the function is all real numbers, except multiples of $\pi\{\dots -2\pi, -\pi, 0, \pi, 2\pi\dots\}$. The range is all real numbers greater than or equal to 1, as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where sine is defined (other than the points at the top and bottom of the sine curve).



Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Many students are reminded of parabolas when they look at the half-period of the cosecant graph. While they are similar in that they each have a local minimum or maximum and they have the same beginning and ending behavior, the comparisons end there. Parabolas are not restricted by asymptotes, whereas the cosecant curve is.

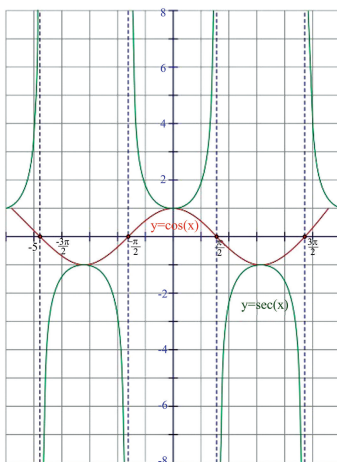
Secant

Secant is the reciprocal of cosine, or $\frac{1}{\cos x}$. Therefore, whenever the cosine is zero, the secant is going to have a vertical asymptote because it will be undefined. It also has the same sign as the cosine function in the same quadrants. Here is the graph.



The period of the function is 2π , just like cosine. The domain of the function is all real numbers, except multiples

of π starting at $\frac{\pi}{2} \left\{ \dots, -\frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \right\}$. The range is all real numbers greater than or equal to 1 as well as all real numbers less than or equal to -1. Notice that the range is everything *except* where cosine is defined (other than the tops and bottoms of the cosine curve).



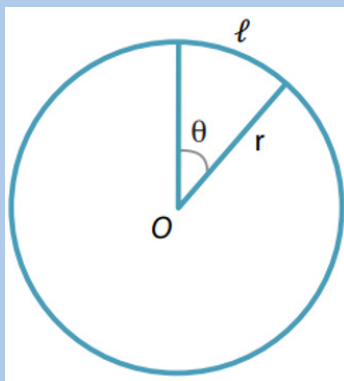
Parabolic usually refers to something in a shape of a parabola. It is also the adjectival form of parable.

Notice again the reciprocal relationships at 0 and the asymptotes. Also look at the intersection points of the graphs at 1 and -1. Again, this graph looks **parabolic**, but it is not. The table below summarizes the functions with their domains and ranges:

Function	Domain	Range
Sin x	all reals	{y: -1 ≤ y ≤ 1}
Cos x	all reals	{y: -1 ≤ y ≤ 1}
Tan x	$\{x: x \neq n \times \frac{\pi}{2}, \text{ where } n \text{ is any odd integer}\}$	all reals
Csc x	$\{x: x \neq n\pi, \text{ where } n \text{ is any integer}\}$	{y: y ≥ 1 or y ≤ -1}
Sec x	$\{x: x \neq n \times \frac{\pi}{2}, \text{ where } n \text{ is any odd integer}\}$	{y: y ≥ 1 or y ≤ -1}
Cot x	$\{x: x \neq n\pi, \text{ where } n \text{ is any integer}\}$	all reals

CONCEPT OF ARC LENGTHS, SECTORS AND SEGMENTS

Measuring angles in radians enables us to write down quite simple formulas for the arc length of part of a circle and the area of a sector of a circle. In any circle of radius r , the ratio of the arc length ℓ to the circumference equals the ratio of the angle θ subtended by the arc at the center and the angle in one revolution.



Thus, measuring the angles in radians,

$$\frac{\ell}{2\pi r} = \frac{\theta}{2\pi}$$

$$\Rightarrow \ell = r\theta.$$

It should be stressed again that, to use this formula, we require the angle to be in radians.

SUMMARY

- In general, the degree measure is used in earlier classes of mathematics since it is easy to understand and remember as this involves use of a protractor to measure the angle.
- Circular functions are the functions whose domain are the set of numbers that correspond to the measures of angles of with respect to the trigonometric functions.
- A radian is the angle subtended at the center of a circle by an arc of the circle equal in length to its radius.
- Radians are the standard mathematical way to measure angles. One radian is equal to the angle created by taking the radius of a circle and stretching it along the edge of the circle.
- Using radian measure allows us to write trigonometric functions as functions not only of angles but also of real numbers in general.
- Any angle has infinitely many conterminal angles because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location.
- The unit circle has as its center the origin point of the Cartesian coordinates x and y , and has a radius 1. The trigonometric functions are also called circular functions, because they describe relationships between angles on the unit circle.
- The trigonometric functions are related to one another because the formula for the unit circle is $x^2 + y^2 = 1$.
- The cosine of an angle is the same as the sine of its complementary angle.
- The name of the tangent function comes from the tangent line of a circle. This is a line that is perpendicular to the radius at a point on the circle so that the line touches the circle at exactly one point.

MULTIPLE CHOICE QUESTIONS

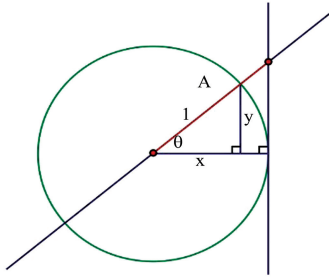
1. 1 radian is
 - a. $54^{\circ}48'$
 - b. $57^{\circ}16'$
 - c. 180°
 - d. $17^{\circ}46'$
2. If angle of arc is 60° and the length of arc is 20 cm. Find the radius of the circle from which arc is intercepted.
 - a. 18.08 cm
 - b. 17.07 cm
 - c. 19.09 cm
 - d. 18 cm
3. If length of arc is 40 cm and radius of circle of arc is 10 cm then find the angle made by the arc.
 - a. 720°
 - b. $240^{\circ}51'53''$
 - c. $229^{\circ}10'59''$
 - d. $233^{\circ}11'48''$
4. If in two circles, arcs of the same length subtend angles 45° and 60° at Centre, find the ratio of their radii.
 - a. 2:3
 - b. 2:5
 - c. 3:4
 - d. 4:3
5. What is the relationship between the sine and cosine functions for any angle θ in a right triangle?
 - a. $\sin(\theta) = 1/\cos(\theta)$
 - b. $\sin(\theta) = \cos(\theta)$
 - c. $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$
 - d. $\sin(\theta) = \tan(\theta)$

Answer

1. (b) 2. (c) 3. (c) 4. (d) 5. (a)

REVIEW QUESTIONS

1. What do you understand by radian?
2. How do you measure radian of an angle?
3. Convert 10 radians to degrees.
4. What is the measure (in radians) of a central angle θ that intercepts an arc of length 6 centimeters on a circle with radius 2 meters?
5. In a circle with radius 10 centimeters, an arc is intercepted by a central angle with measure $\frac{7\pi}{4}$. Find the arc length.
6. Evaluate $\cos\left(\frac{5\pi}{4}\right)$ exactly.
7. Show that side A (in orange) in this drawing is equal to $\sec \theta$. Use similar triangles in your proof.

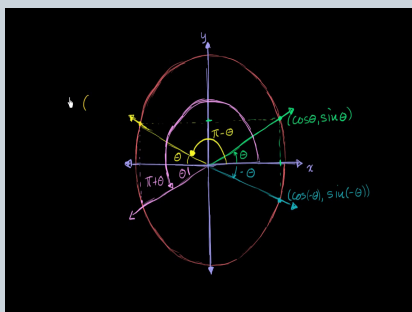


8. Explain the three reciprocal functions.

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GRAPHS OF THE CIRCULAR FUNCTIONS



Circular functions, are functions of an angle. They relate the angles of a triangle to the lengths of its sides. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.

After studying this chapter, you will understand the core concepts of:

- Graphing Trig Functions
- Phase Shift

INTRODUCTION

The circular functions are a group of trigonometric functions that relate angles to the coordinates of points on the unit circle. The most commonly used circular functions are sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot). These functions are widely used in mathematics, physics, engineering, and various other fields to model periodic phenomena.

When graphed, the sine and cosine functions create sinusoidal curves that repeat every 360 degrees or 2π radians. The unit circle is often used as a reference to understand the relationships between angles and coordinates.

Here's a brief overview of the graphs of the sine and cosine functions:

Sine Function (sin):

- The sine function oscillates between -1 and 1.
- The graph starts at the origin (0, 0) and moves upward, reaching its maximum value of 1 at $\pi/2$ radians (90 degrees).
- The graph then crosses the x-axis again at π radians (180 degrees), reaching its minimum value of -1.
- The pattern repeats every 2π radians (360 degrees).

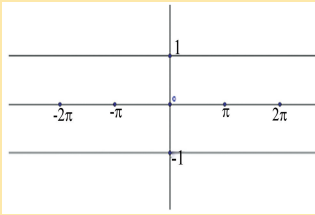
Cosine Function (cos):

- The cosine function also oscillates between -1 and 1.
- The graph starts at its maximum value of 1 at 0 radians (0 degrees).
- It crosses the x-axis at $\pi/2$ radians (90 degrees) and reaches its minimum value of -1 at π radians (180 degrees).
- The graph's pattern also repeats every 2π radians (360 degrees).

The tangent function (tan) is defined as the ratio of sine to cosine and can be graphed as well. However, its graph has vertical asymptotes where the cosine is zero (at odd multiples of $\pi/2$ radians or 90-degree intervals), resulting in an oscillating curve with rapid changes.

The graphs of cosecant (csc), secant (sec), and cotangent (cot) can be derived from the sine, cosine, and tangent functions by taking their reciprocal values. They also exhibit periodic behavior and have their own unique characteristics.

It's important to note that these graphs are essential for understanding waveforms, vibrations, and other periodic phenomena. They are also used extensively in calculus, geometry, and physics to solve various types of problems involving angles and periodic functions.



CORE CONCEPT: GRAPHING TRIG FUNCTIONS

The graphs of trigonometric functions have several properties to elicit. To be able to graph these functions by hand, we have to understand them.

Let's graph the trigonometric functions on the coordinate system. The x values will be the angles (in radians – that's the way it's done), and the y values will be the trig value (like $\sin$, $\cos$, and $\tan$).

The $\sin$ and $\cos$ (and $\csc$ and $\sec$) functions start repeating after 2π radians, and it turns out that the $\tan$ and $\cot$ functions start repeating again after only π radians. The reason $\tan$ (and $\cot$) repeat after only π radians is because, when dividing $\sin$ and $\cos$ to get $\tan$, we get the same values in quadrants III as IV as we do for quadrants I and II.

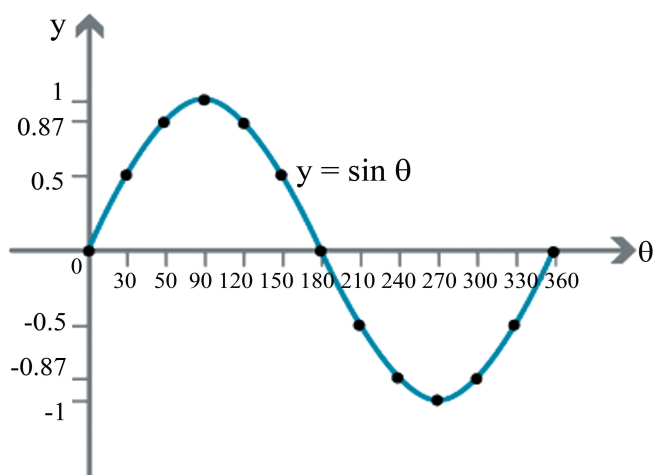
A complete repetition of the pattern of the function is called a cycle and the period is the horizontal length of one complete cycle. Thus the period of the $\sin$, $\cos$, $\csc$, and $\sec$ graphs is 2π radians, and the period for the $\tan$ and $\cot$ graphs is π radians.

Starting and stopping points may be changed, as long the graph covers one complete cycle (period).

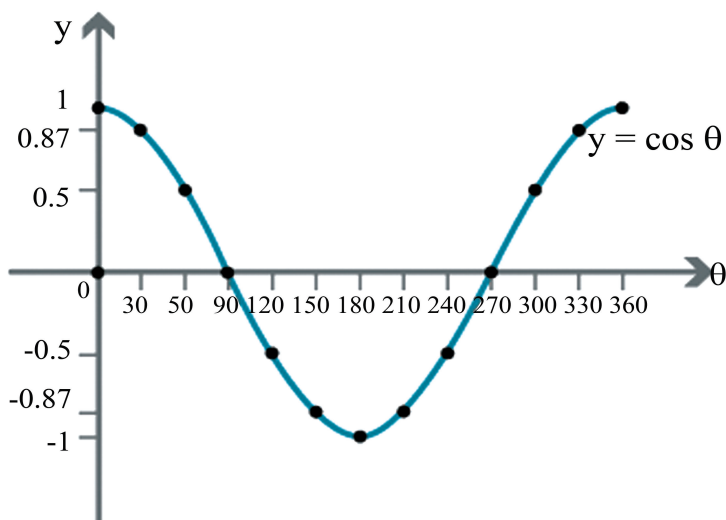
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$y = \sin(x)$	$y = \cos(x)$	$y = \tan(x)$	$y = \csc(x)$	$y = \sec(x)$	$y = \cot(x)$																																																																								
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Graphing Trigonometric Functions

In the module trigonometric functions, we drew the graphs of the sine and cosine functions, marking the θ -axis in degrees. Using $5\sin 30^\circ = 0.5$ and $60^\circ \approx 0.87$, we drew up a table of values and plotted them.



These graphs are often referred to by physicists and engineers as **sine waves**.



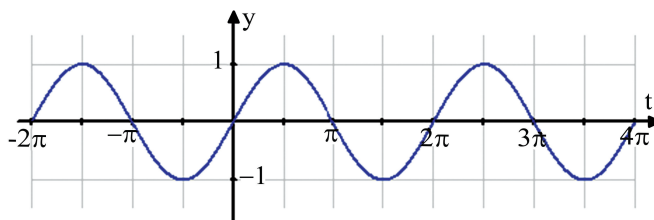
Function of several variables is a function that depends on several arguments.

From now on, we will use radians as the unit on the θ -axis and so we have the following graphs for sine and cosine.

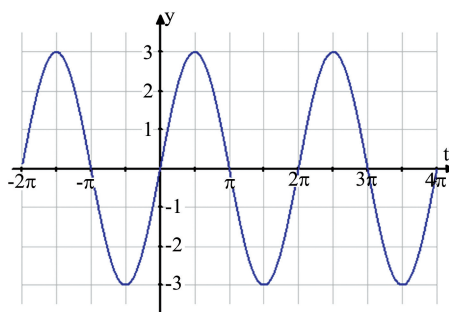
You have already learned the basic trig graphs. But just as you could make the basic quadratic, $y = x^2$, more complicated, such as $y = -(x + 5)^2 - 3$, so also trig graphs can be made more complicated. We can transform and translate trig functions, just like you transformed and translated other **functions** in algebra.

Let's start with the basic sine function, $f(t) = \sin(t)$. This function has an amplitude of 1 because the graph goes one unit up and one unit down from the midline of the graph. This function has a period of 2π because the sine wave repeats every 2π units.

The graph looks like this:



Now let's look at $g(t) = 3\sin(t)$:



Do you see that this second graph is three times as tall as was the first graph? The amplitude has changed from 1 in the first graph to 3 in the second, just as the multiplier in front of the sine changed from 1 to 3. This relationship is always true: Whatever number A is multiplied on the trig function gives you the amplitude (that is, the “tallness” or “shortness” of the graph); in this case, that amplitude number was 3.

- What is the amplitude of $y(t) = 0.5\cos(t)$?

For this function, the value of the amplitude multiplier A is given by 0.5, so the function will have an amplitude of:

$$0.5 = 1/2$$

- What is the amplitude of $y(x) = -2\cos(x)$?

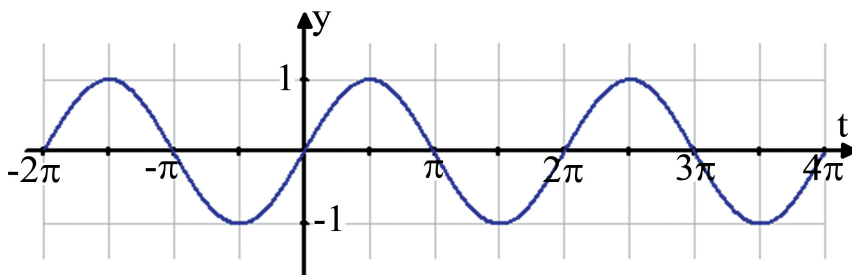
For this function, the value of the amplitude multiplier A is -2 , so the amplitude is:

2

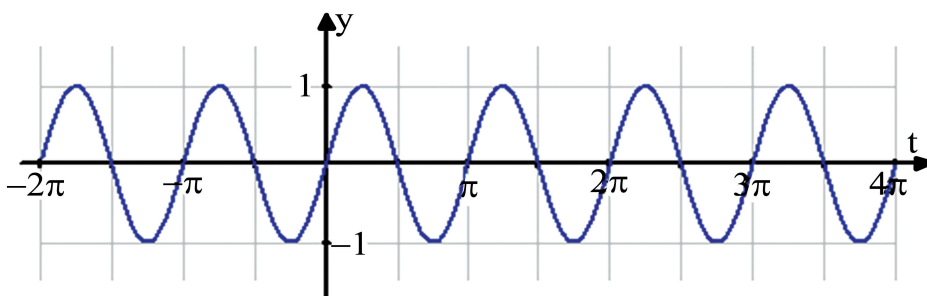
...and, by the way, the graph would also be flipped upside down, because of the “minus” sign.

Technically, the amplitude is the absolute value of whatever is multiplied on the trig function. The amplitude just says how “tall” or “short” the curve is; it’s up to you to notice whether there’s a “minus” on that multiplier, and thus whether or not the function is in the usual orientation, or upside-down.

Recall the first graph, being the “regular” sine wave:



Now let's look at $h(t) = \sin(2t)$:



Do you see how this third graph is squished in from the sides, as compared with the first graph? Do you see that the sine wave is cycling twice as fast, so its period is only half as long? This relationship is always true: Whatever value B is multiplied on the variable (inside the trig function), you use this value to find the period ω (“omega”, not “double-u”) of the trig function, according to this formula:

General period formula:

$$\omega = \frac{\langle \text{regular period} \rangle}{|B|}$$

For sines and cosines (and their reciprocals), the “regular” period is 2π , so their formula is:

Period formula for sines and cosines:

$$\omega = \frac{2\pi}{|B|}$$

For tangents and cotangents, the “regular” period is π , so their formula is:

Period formula for tangents and cotangents:

$$\omega = \frac{\pi}{|B|}$$

In the sine wave graphed above, the value of the period multiplier B was 2. (Sometimes the value of B inside the function will be negative, which is why there are absolute-value bars on the denominator.) As a result, its period was $2\pi/2 = \pi$.

- What is the period of $f(t) = \cos(3t)$?

The formula for sines and cosines says that the regular period is 2π . In $\cos(3t)$, $B = 3$, so this function will have a period of:

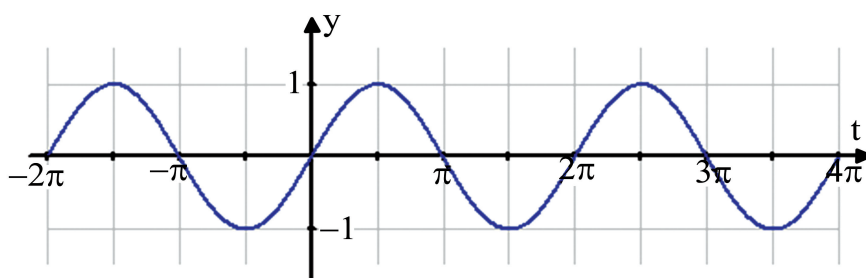
$$\frac{2\pi}{3} = \left(\frac{2}{3}\right)\pi$$

- What is the period of $g(x) = \tan(x/2)$?

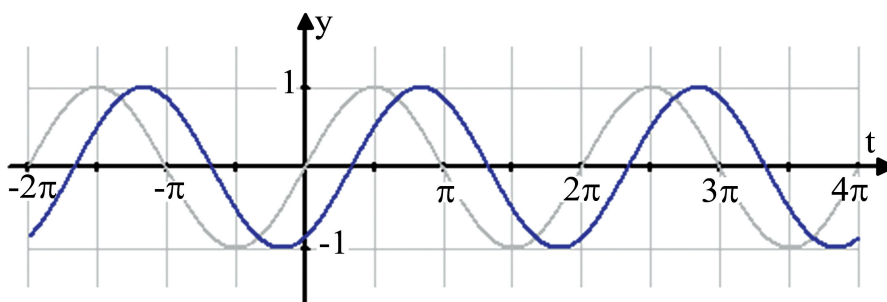
The formula for tangents and cotangents says that the regular period is π . For $\tan(t/2)$, the value of B is $1/2$, so the function will have a period of:

$$\frac{\pi}{\left(\frac{1}{2}\right)} = 2\pi$$

Recall again the first graph, being the “regular” sine wave:



Now let's look at $j(t) = \sin(t - \pi/3)$:



Do you see that the graph (shown in blue on the graph above) is shifted over to the right by $\pi/3$ units from the regular graph (shown in gray)? This relationship is always true: If the argument of the function (the thing you are plugging in to the function) is of the form “(variable) – (number) = (variable) – C ”, then the graph is shifted to the *right* by that (number) of units (that is, by C units); if the argument is of the form “(variable) + (number) = (variable) + C ”, then the graph is shifted to the *left* by that (number) of units (again, by C units). This right- or left-shifting is called “phase shift”.

$$y(t) = \cos\left(x + \frac{\pi}{4}\right)?$$

- What is the phase shift of

Inside the argument (that is, inside the parentheses of the function), a $\pi/4$ is added to the variable. This means that $C = \pi/4$. Because this value is *added* to the variable, then the shift is to the *left*. Then the phase shift is:

to the left by $\frac{\pi}{4}$

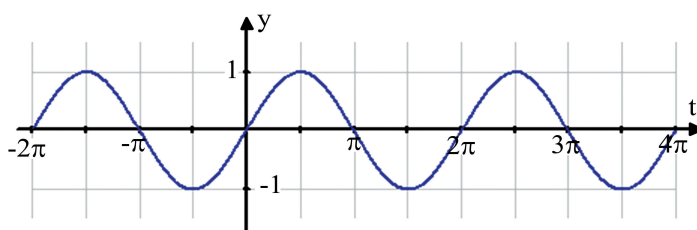
$$f(t) = \tan\left(t - \frac{2\pi}{3}\right)$$

- What is the phase shift of

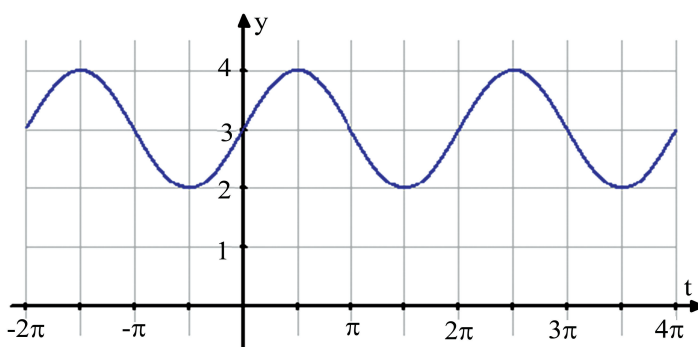
The number C inside with the variable is $2\pi/3$, so this will be the phase shift. This number is *subtracted* from the variable, so the shift will be to the *right*.

to the right by $\frac{2\pi}{3}$

Let's recollect again the graph of the “regular” sine wave:



Now let's look at $k(t) = \sin(t) + 3$:



Do you see how the graph was shifted up by three units? This relationship is always true: If a number D is added outside the function, then the graph is shifted up by that number of units; if a number D is subtracted, then the graph is shifted down by that number of units.

- By what amount is the graph of $h(t) = \cos(t) - 2$ shifted, and in which direction?

The trig-function part is the $\cos(t)$; the up-or-down shifting part is the $D = -2$. There's nothing else going on inside of the function, nor multiplied in front of it, so this is the regular cosine wave, but it's:

shifted downward two units

- By what amount is the graph of $t(x) = \tan(x) + 0.6$ shifted, and in which direction?

The trig-function part is the $\tan(x)$; the up-or-down shifting part is the $+ 0.6$. So this is the regular tangent curve, but:

shifted upward by $6/10 = 3/5$ of a unit.

Putting it all together in terms of the sine wave, we have the general sine function:

$$F(t) = A\sin(Bt - C) + D$$

...where $|A|$ is the amplitude, B gives you the period, D gives you the vertical shift (up or down), and C/B is used to find the phase shift.

Why don't we just use C for the phase shift?" Because sometimes more involved stuff is going on inside the function. Remember that the phase shift comes from what is added or subtracted *directly* to the variable. If the variable is not alone (that is, if there's something multiplied directly on it), then there's another step to follow.

For instance, if you have something like:

$$y = \sin(2t - \pi)$$

...the phase shift is *not* π units! Instead, you first have to isolate what's happening to the variable by factoring, as so:

$$y = \sin\left(2\left(t - \frac{\pi}{2}\right)\right)$$

Now you can see that the phase shift will be $\pi/2$ units, not π units. So the phase shift, as a formula, is found by dividing C by B .

For $F(t) = A f(Bt - C) + D$, where $f(t)$ is one of the basic trig functions, we have:

- the amplitude is $|A|$
- the period is $\omega = \frac{\langle \text{regular period} \rangle}{|B|}$
- the phase shift is $\frac{C}{B}$
- the vertical shift is D

Find the amplitude, period, phase shift, and vertical shift of $s(t) = -2.5\tan(4t - 3\pi) - 4$

The **amplitude** is given by the multiplier on the trig function. In this case, there's a -2.5 multiplied directly onto the tangent. This is the " A " from the formula, and tells me that the amplitude is 2.5. (If you were to be graphing this, you would need to note that this tangent's graph will be upside-down, too.)

Amplitude is the distance your function reaches above the midline. The amplitude really only applies to the sine and cosine functions.

The regular period for tangents is π . In this particular function, there's a 4 multiplied on the variable, so $B = 4$. Plugging into the period formula, you get $4/\pi$.

To find the phase shift, you need to isolate the variable with the shift value, so you need to factor out the 4(also known as "C") that's multiplied on the variable. The factorization is:

$$4t - 3\pi = \left(4t - \frac{3\pi}{4} \right)$$

Then the phase shift is $3/4\pi$. Because the shift value is *subtracted* from the variable, the shift is *to the right*.

(I could also have used the simpler method, directly from the formula, of dividing C by B. This would have given me the same value, but more quickly and with less chance of error in the factorization. Try it both ways yourself, and figure out which one you like better. Then practice it well before the next test!)

The vertical shift comes from the value entirely outside of the trig function; namely, the outer 4 (also known as "D", from the formula). Because this 4 is *subtracted* from the tangent, the shift will be four units *downward* from the usual center line, the x -axis.

- amplitude: 2.5
- period: $\frac{\pi}{4}$
- phase shift: to the right by $\frac{3}{4}\pi$
- vertical shift: downward by 4

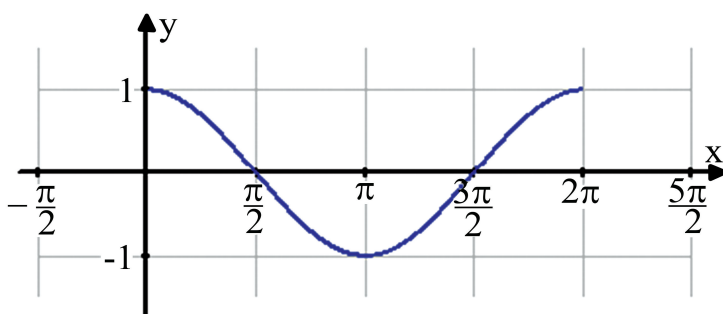
Graphing Trigonometric Functions: Examples

If you know the basic graphs, then the more-complicated graphs can be fairly easy to draw. Here's how it works:

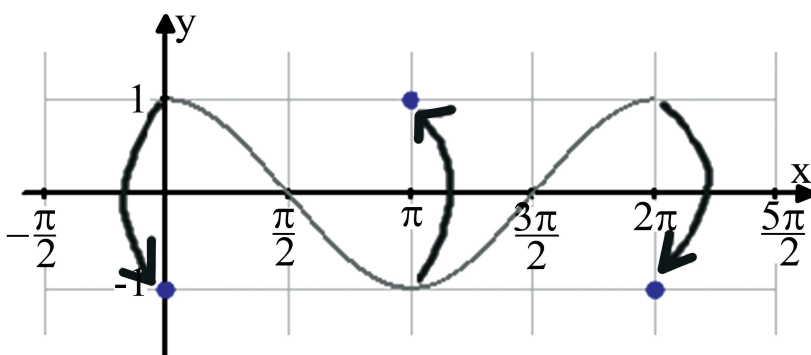
- Graph one period of $s(x) = -\cos(3x)$

The "minus" sign tells me that the graph is upside down. Since the multiplier out front is an "understood" -1 , the amplitude is unchanged. The argument (the $3x$ inside the cosine) is growing three times as fast as usual, because of the 3 multiplied on the variable, so the period is one-third as long. The period for this graph will be $(2/3)\pi$.

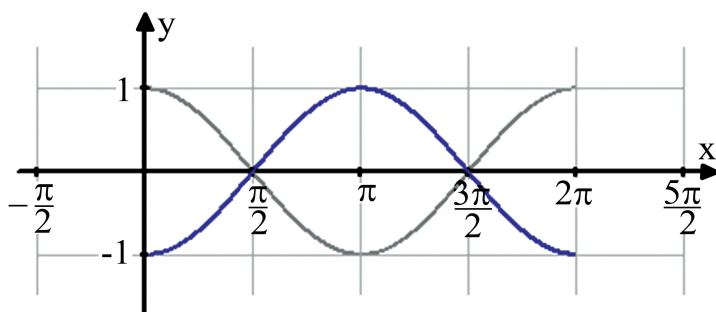
Here is the regular graph of cosine:



You need to flip this upside down, so you will swap the +1 and -1 points on the graph:



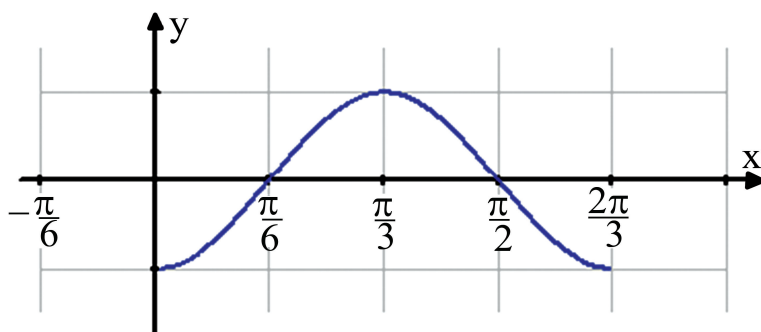
...and then you will fill in the rest of the graph. (The original, “regular”, graph is shown in gray below; my new, flipped, graph is shown in blue.)



That takes care of the amplitude. Now you need to change the period.

Rather than trying to figure out the points for the graph on the regular axis, you will instead re-number the axis, which is a lot easier. The regular period is from 0 to 2π , but this graph’s period goes from 0 to $(2\pi)/3$. Then the midpoint of the period is going to be $(1/2)(2\pi)/3 = \pi/3$, and the zeroes will be midway between the peaks (the high points) and the troughs (the low points). So you will erase the x -axis values from the regular graph, and re-number the axis.

The following is my final (hand-in) graph:

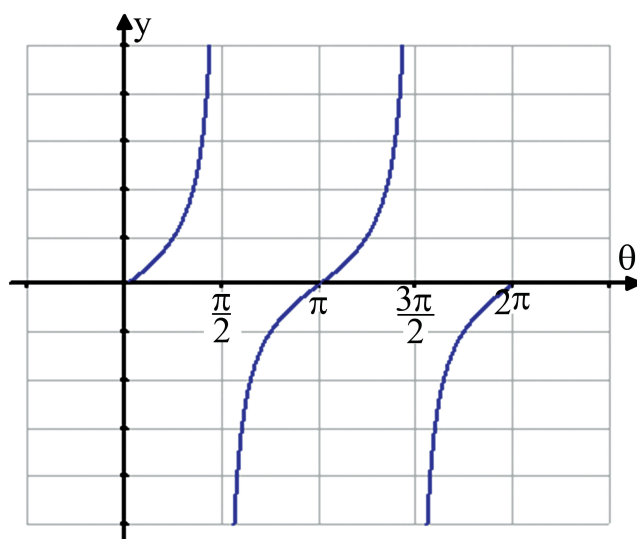


Notice how you changed the axis instead of the graph. You will quickly get pretty good at drawing a regular sine or cosine, but the shifted and transformed graphs can prove difficult. Instead of trying to figure out all of the changes to the graph, just tweak the axis system.

The next example shows how to move things around for a vertical shift.

- Graph at least one period of $f(\theta) = \tan(\theta) - 1$

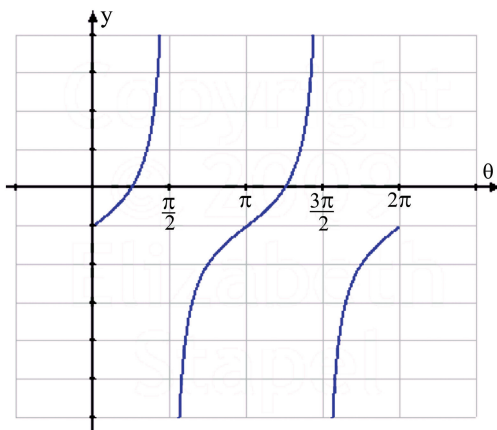
The regular tangent looks like this:



The graph for $\tan(\theta) - 1$ is the same shape as the regular tangent graph, because nothing is multiplied onto the tangent.

But this graph is shifted down by one unit. In other words, instead of the graph's midline being the x -axis, it's going to be the line $y = -1$.

Rather than trying to figure out the points for moving the tangent curve one unit lower, you will just erase the original horizontal axis and re-draw the axis one unit higher. Then my final (hand-in) graph looks like this:



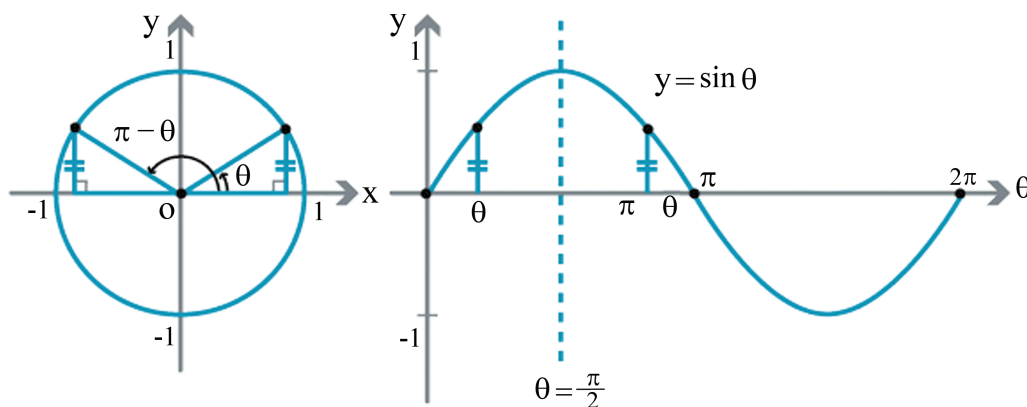
You may, at first, want to use scratch-paper for the various changes (flipping graphs upside-down, moving axes up and down, changing the measurements on the x -axis, etc.), so your hand-in homework is not full of erasures. But get used to working neatly, from start to finish, on the hand-in sheet, so your work on the next test will be acceptable.

Symmetries of the sine Graph

The graph of $y = \sin \theta$ has many interesting symmetries.

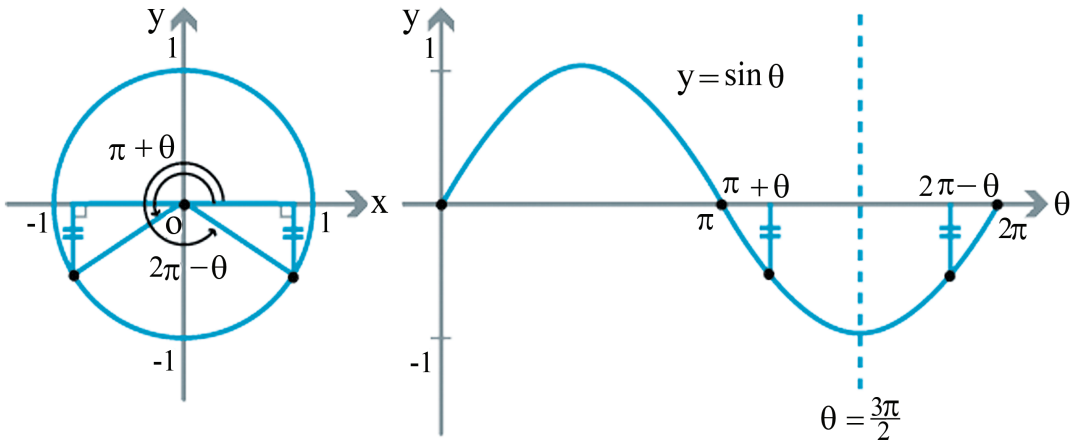
The model employing the unit circle helps to elucidate these. The sine of the angle θ is represented by the y -value of the point P on the unit circle. Since $\sin \theta = \sin(\pi - \theta)$, we can mark two equal intervals on the following sine graph.

Start by drawing lightly in pencil, and have a good eraser (like from an office supply or craft store). Be sure you are using a ruler for your final drawing. Also, it can be helpful to use a regular pencil for the temporary “regular” graph, but then use colored pencils for your final version.

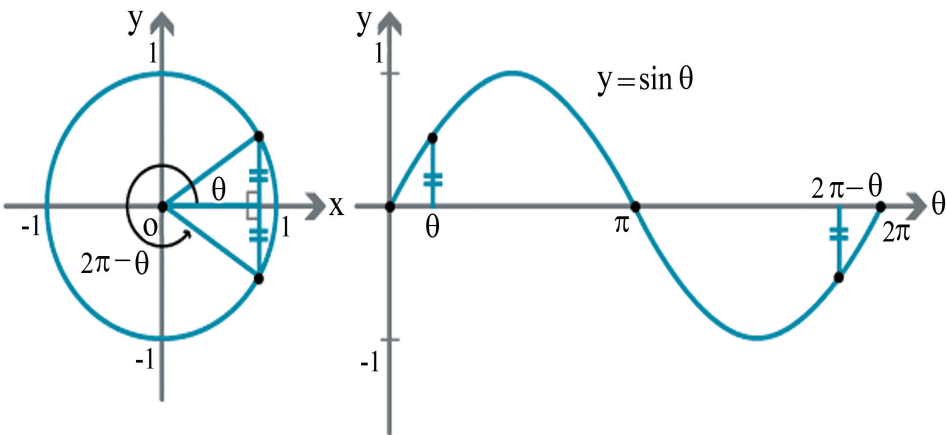


Hence, between 0 and π , the graph of $y = \sin \theta$ is symmetric about $\theta = \frac{\pi}{2}$.

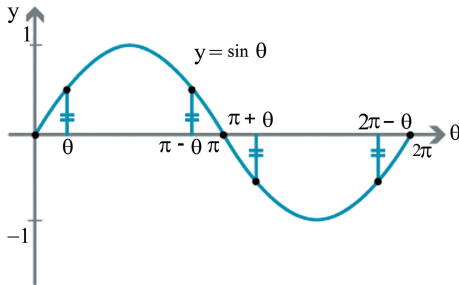
Similarly, between π and 2π , the graph is symmetric about $\theta = \frac{3\pi}{2}$.



Finally, the graph possesses a rotational symmetry about $\theta = \pi$ as the following diagram demonstrates.



All these observations are summarized by the following diagram. This symmetry diagram illustrates the related angle and the quadrant sign rules, as well as the symmetries discussed above.



Symmetries of the sine graph.

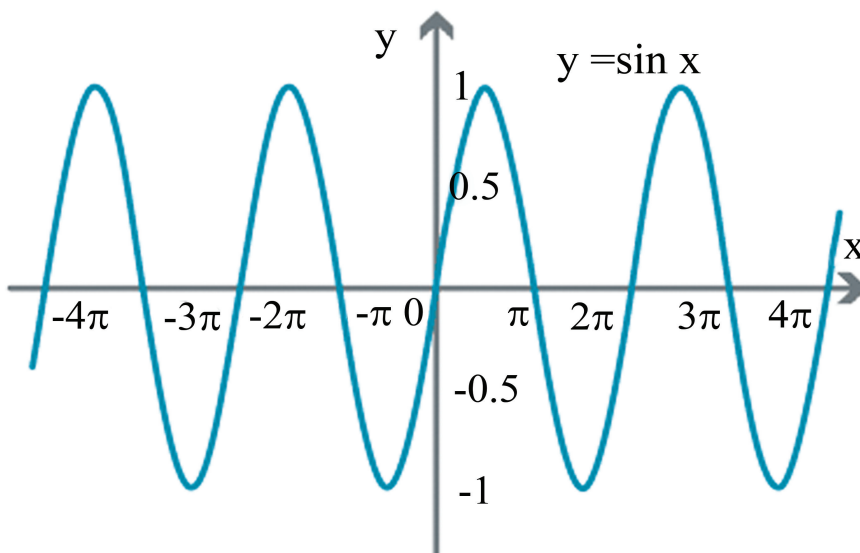
The Extended sine Graph

The values of sine repeat as we move through an angle of 2π , that is,
 $\sin(2\pi+x)=\sin x$.

We say that the function $y=\sin x$ is periodic with period 2π .

Thus, the graph may be drawn for angles greater than 2π and less than 0, to produce the full (or extended) graph of $y=\sin x$.

The graph of $y=\sin x$ from 0 to 2π is often referred to as a **cycle**.



Note that the extended sine graph has even more symmetries. There is a translation (by 2π) symmetry, a reflection symmetry about any odd multiple of $\frac{\pi}{2}$ and a rotational symmetry about any even multiple of $\frac{\pi}{2}$.

Types of Trigonometric Function

Graphing a trigonometric function is actually pretty easy if you know what numbers to look at. Recall that a trigonometric function ('trig function') is a mathematical function of an angle. There are three basic trig functions:

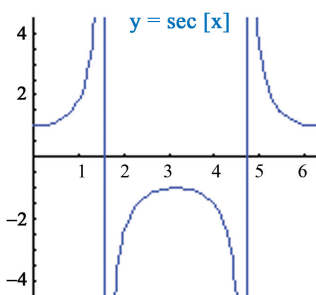
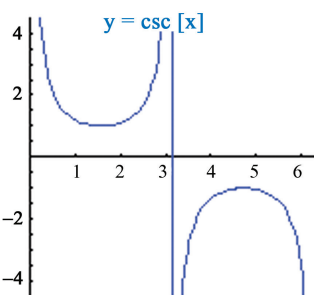
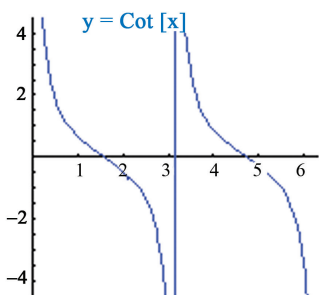
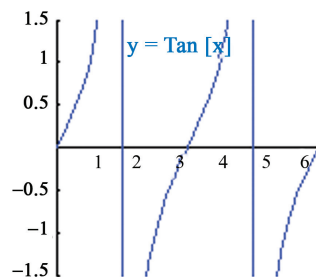
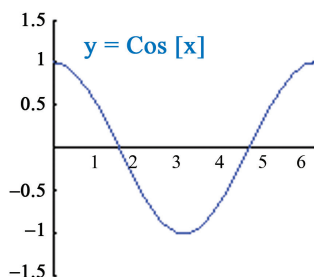
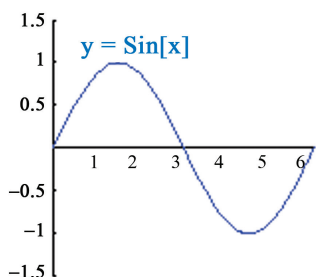
- the sine function
- the cosine function
- the tangent function

There's also three inverse functions:

- the cosecant function - the reciprocal of the sine function
- the secant function - the reciprocal of the cosine function

- The cotangent function - the reciprocal of the tangent function.

To graph these functions, it's helpful to know what their basic graphs look like without any additional numbers changing the graph:



Now you can use the properties of trigonometric functions to help you graph any one. This is because all trigonometric functions follow the same rules.

Take the standard trigonometric function:

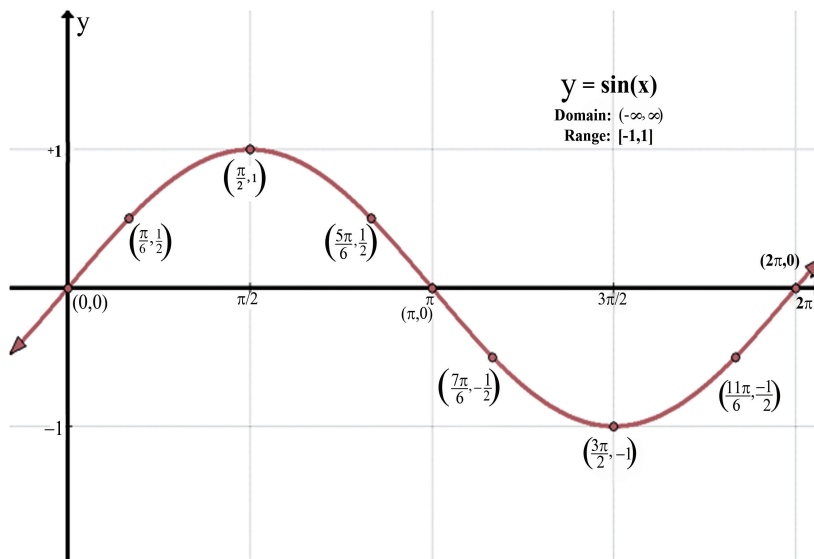
$$f(x) = A \text{trig}(Bx - C) + D$$

- *trig* stands for basic trigonometric function, which could be either cosine, sine, or tangent.
- A is amplitude
- The phase shift is given by C / B
- The vertical shift is given by D

Let's take a look at how these values change the basic graphs.

Graph of Sin Function

Here's what a **sin function** graph looks like; notice that the **domain** is $(-\infty, \infty)$, and the **range** is $[-1, 1]$.



Note again that the graph starts repeating itself (cycles) every 2π radians (the x); this is called the period of the graph. Note that this is one complete revolution around the unit circle.

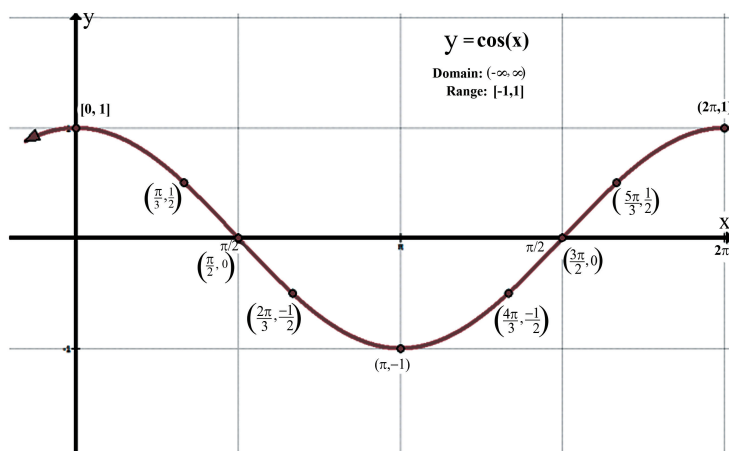
Notice that the y intercept is $(0, 0)$, and the x -intercepts (zeros) are multiples of π ; we can write this as $(\pi k, 0)$, where $k \in \mathbb{I}$ (k is in the set of Integers).

The sin function is an odd function; we learned about Even and Odd Functions here. Thus the sin graph is symmetrical about the origin $(0, 0)$, meaning that if (x, y) is a point on the function (graph), then so is $(-x, -y)$. It also means that for the sin graph,

$$f(-x) = -f(x). \text{ Let's try this: } \left(-\frac{\pi}{2}\right) = -1 = -\sin\left(\frac{\pi}{2}\right).$$

Graph of Cos Function

Now let's graph the cos function; notice again that the domain is $(-\infty, \infty)$, and the range is $[-1, 1]$.



Note again that the graph starts repeating itself (cycles) every 2π radians (the x); this is called the period of the graph. Note that this is one complete revolution around the unit circle.

Notice that the y intercept is $(0, 1)$, and the x intercepts are multiples of $\frac{\pi}{2}$; we can write this as $\left(\frac{\pi}{2} + \pi k, 0\right)$, where $k \in I$ (k is in the set of Integers).

The $\cos$ function is an even function; we learned about Even and Odd Functions here. Thus the $\cos$ graph is symmetrical about the y axis, meaning that if (x, y) is a point on the function (graph), then so is $(-x, y)$. It also means that for the $\cos$ graph,

$$f(-x)=f(x). \text{ Let's try this: } \left(-\frac{\pi}{3}\right) = -\frac{1}{2} = \cos\left(\frac{2\pi}{3}\right)$$

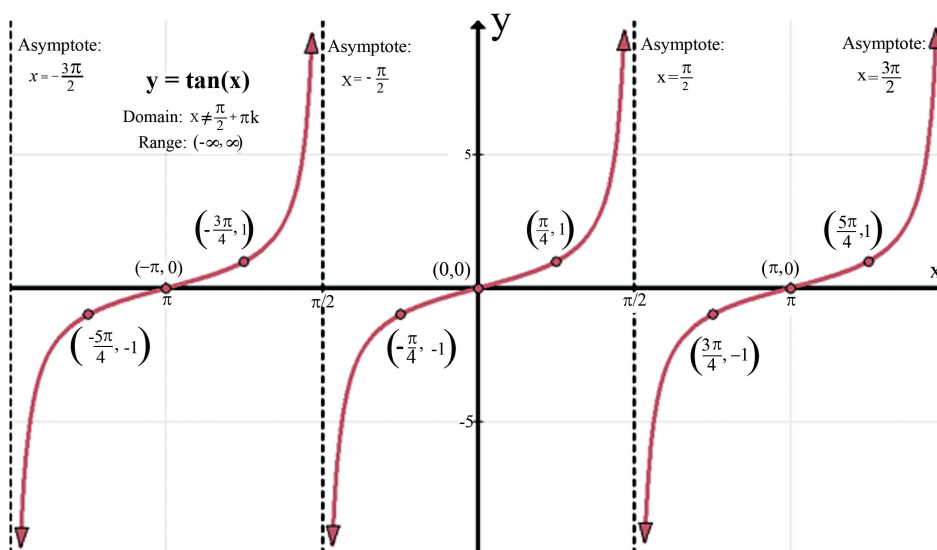
The graph of $\tan x$

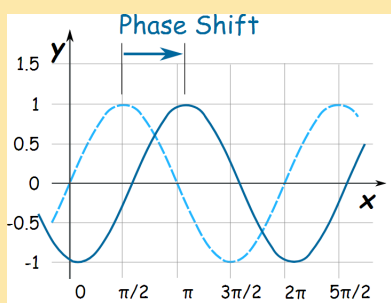
The function $\tan x$ has a very different kind of graph to those for the sine and cosine functions. The period of $\tan x$ is π , rather than 2π , and the amplitude is not defined. The

tangent function is not defined at $x = \pm \frac{\pi}{2}$, nor at any odd integer multiple of these values.

As x approaches $\frac{\pi}{2}$ from the left, the value of $\tan x$ increases without bound. Since $\tan$

is an odd function, as x approaches $-\frac{\pi}{2}$ from the right, $\tan x$ decreases without bound.





CORE CONCEPT: PHASE SHIFT

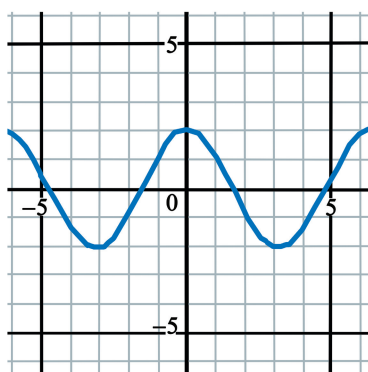
Phase shift is a common term in the study of waveforms and communication of signals. It refers to the displacement of two signals when propagated in a time domain. This displacement can be caused by a signal processing device such as an electronic amplifier or a low- or high-pass filter that performs

certain operations on the signal, causing the output signal phase to shift from its original input signal phase.

The phase shift tells you where your period begins. Basic functions all begin at $x = 0$. A phase shift moves this value either to the left or right. A phase shift of 2 means you are shifting your period 2 spaces to the right. You get a phase shift of 2 whenever the ratio of your C to B is 2, so $C / B = 2$ (for example, if $C = 2$ and $B = 1$).

Remember that in the standard function, C is being subtracted, so if you see a minus, C is actually positive. But if you see a plus, then C is actually negative. A negative phase shift moves the graph to the left.

So, the function $f(x) = \sin(x - 2)$ moves the graph of $\sin(x)$ two spaces to the right.

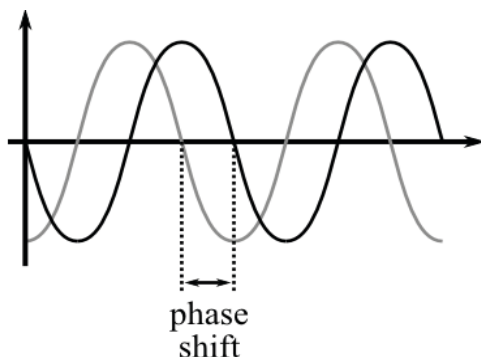


Frequency is the number of occurrences of a repeating event per unit of time. It is also referred to as temporal frequency, which emphasizes the contrast to spatial frequency and angular frequency.

It is important to understand that phase shift does not change the **frequency** of the signal. Two signals having a phase shift may or may not be of the same frequency. Phase shift simply means that the two signals are at different points of their cycle at a given time. Phase shift is measured as the angle (in degrees or radians) between two points on a circle at

the same time, demonstrating the progress of each wave through its cycle. Phase shift is more easily observed in sine waves where there is a single fundamental frequency and no harmonics.

With regard to wave motion, a phase shift represents the amount a wave has shifted horizontally from the original wave. Phase shifts are typically measured in degrees where a complete cycle is 360° . In the diagram below, the second wave is shifted by the specified amount from the original wave:



Because the horizontal axis denotes time, a phase shift represents a shift in time from the original wave. The signal in a circuit, *for example*, will have some phase shift between the input and the output. When designing an operational amplifier, for example, engineers prefer to minimize the phase shift as much as possible to avoid unwanted oscillations.

Graph two periods of $\pm \left(\right) \sin \left(\pi \frac{\pi}{2} \right) 3$

The amplitude of this graph is going to be the same as for regular sine waves, because there's an "understood" 1 multiplied on the sine.

But the midline of the graph is going to be at $y=3$ instead of $y=0$ (that is, the x -axis), because of the "+3" at the end of the function.

The regular period for sine waves is 2π , but the variable in this function is multiplied

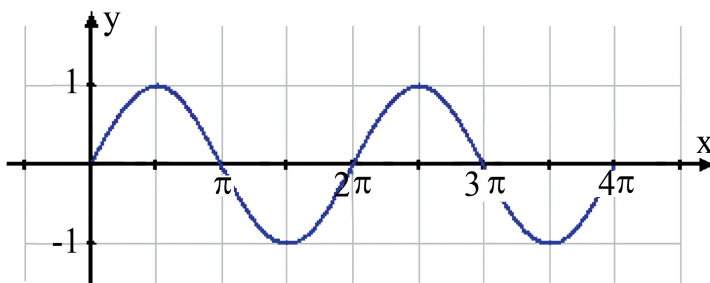
by π ; doing the division, the period of this particular function is going to be $\frac{2\pi}{\pi} = 2$. Since you have to graph "at least two periods" of this function, you will need my x -axis to be at least four units wide.

Now, the new part of graphing: the phase shift. Looking inside the argument, you see that there's something multiplied on the variable, and also that something is added onto it. To figure out the actual phase shift, you will have to factor out the multiplier,

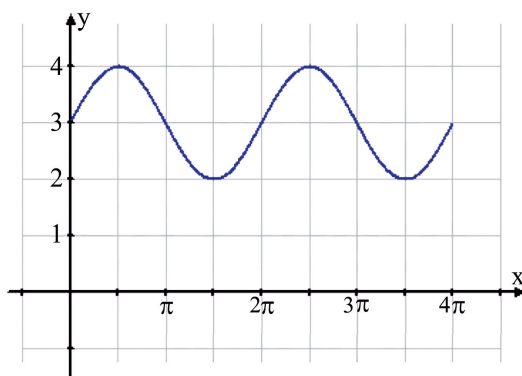
π , on the variable. The argument factors as $\left(\quad - \right)$. Now you can see that there's a

$\frac{1}{2}$ added to the variable, so the graph will be shifted $\frac{1}{2}$ units to the left.

You will start with the usual graph:

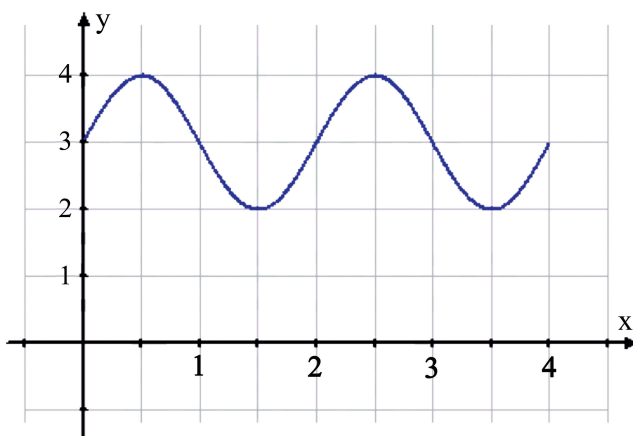


You know that this graph has a vertical shift upwards of three units. But, instead of shifting the graphed sine wave three units up, you will add room underneath my current graph, shift the horizontal axis three units down, and then re-number the y-axis:



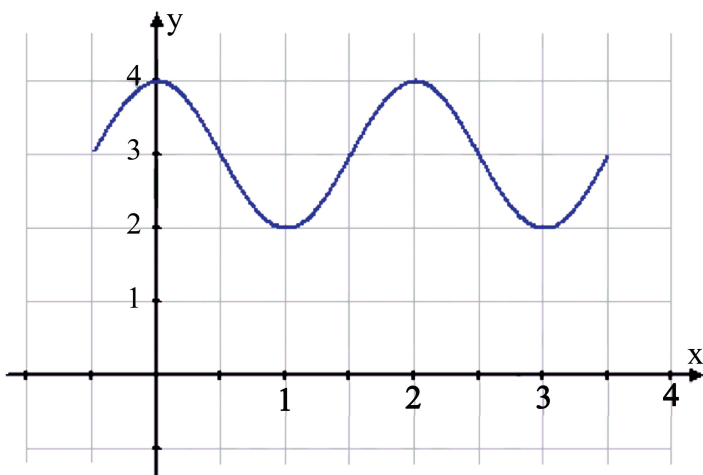
The above is the same graph; all you've really done is moved the x-axis down three units, redrawn and relabeled it, and then renumbered the y-axis. You have not touched the graphed sine wave, drawn in blue.

The regular period for sine waves goes from 0 to 2π ; this one goes from 0 to 2, so you will re-number the x-axis:



All you have done in the above graph is relabel the x -axis from π , 2π , 3π , and 4π to 1, 2, 3, and 4, respectively. You have not touched my blue graphed line.

From the phase-shift computations, you know that the graph is shifted to the left by $\frac{1}{2}$, so you will shift the y -axis to the right by $\frac{1}{2}$ and re-number the x -axis again. This is the last bit of computation, so this is my final graph:



Graph is an abstract data type that is meant to implement the undirected graph and directed graph concepts from mathematics, specifically the field of graph theory.

All you did to get the last graph above was erase the existing y -axis, redraw it half a unit to the right, and then erase and rewrite the labels on the x -axis, again half a unit to the right. You never touched my blue-line graph.

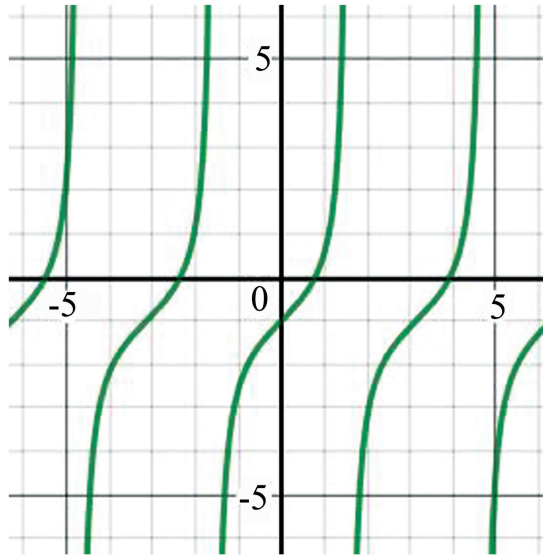
My best advice regarding doing these **graphs** is to practice, practice, and practice. You do not want to freeze or have a “brain-fart” on the test, and this is pretty straightforward once you get the feel of it. So keep doing extra graphing, until you are feeling comfortable and confident in your skills.

The Midline

The midline is the line in the middle of the graph. For plain functions without added values, the midline is the $y = 0$ line. This midline changes if the D value creates a vertical shift.

If $D = 1$, then your graph will be shifted up by 1. If $D = -2$, then your graph will be shifted down by 2.

For example, the graph of $f(x) = \tan(x) - 1$ will be your $\tan(x)$ graph shifted down by 1.



Shifts

Graph: $f(x) = (x - 1)^2 + 4$ This function comes from the base function $f(x) = x^2$. Since 1 is subtracted on the inside, this is a horizontal shift RIGHT 1 unit, and since 4 is added on the outside, this is a vertical shift UP 4 units.

Horizontal and Vertical Shifts

Among the variations on the graphs of the trigonometric functions are shifts--both horizontal and vertical. Such shifts are easily accounted for in the formula of a given function.

Take function f , where $f(x) = \sin(x)$. The graph of $y = \sin(x)$ is seen below.

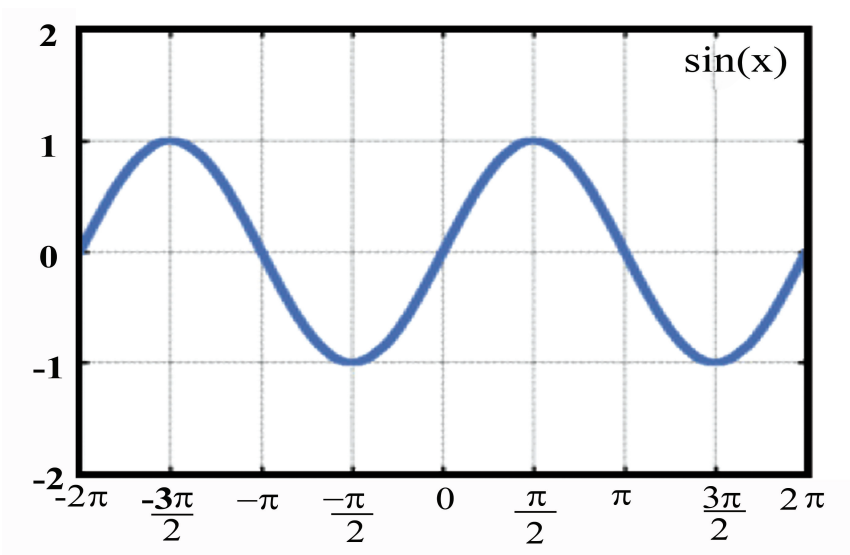


Figure 1: The Graph of $\sin(x)$.

Vertical Shifts

To shift such a graph vertically, one needs only to change the function to $f(x) = \sin(x) + c$, where c is some constant. Thus the y -coordinate of the graph, which was previously $\sin(x)$, is now $\sin(x) + 2$. All values of y shift by two. The graph of $y = f(x) = \sin(x) + 2$ looks like this:

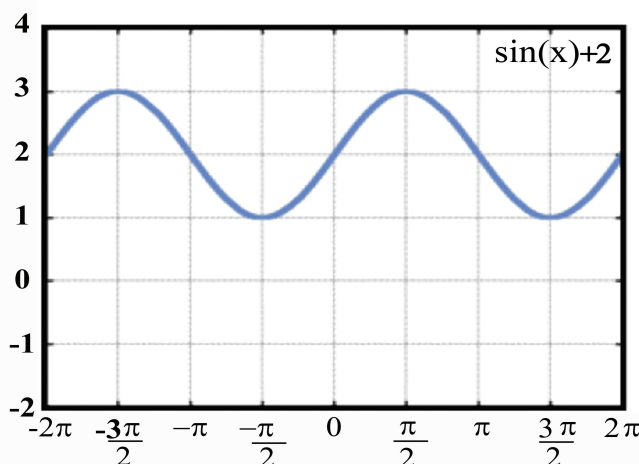


Figure 2: By adding a constant to a function, like sine, the graph is shifted vertically.

Horizontal Shifts

To shift a graph horizontally, a constant must be added to the function within parentheses--that is, the constant must be added to the angle, not the whole function.

Compare the to the graph of $y = f(x) = \sin(x + \frac{\pi}{2})$.

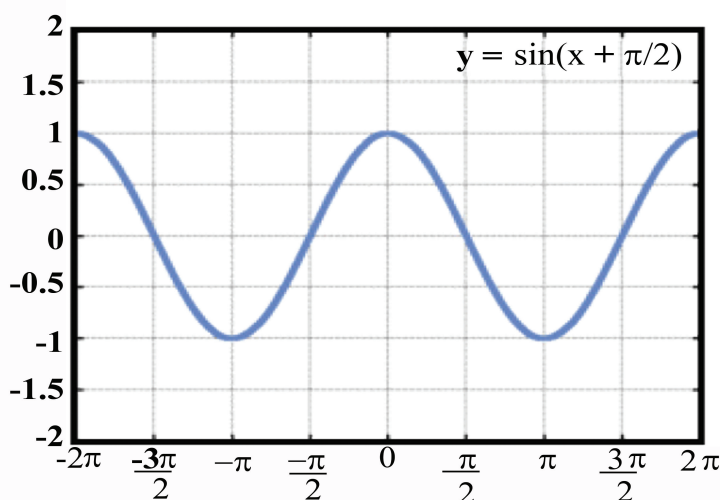


Figure 3: Horizontal shift.

The graph of sine is shifted to the left by $\frac{\pi}{2}$ units. Does it look familiar? The graph is the same as the cosine graph. Sine and cosine are both periodic functions, and have

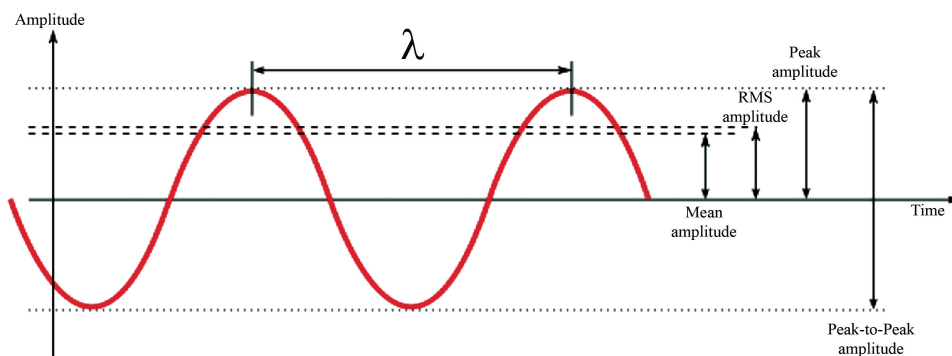
the same domain and range. They are separated only by a phase shift: $\frac{\pi}{2}$. This is the constant that must be added to create the necessary horizontal shift to make the graphs directly overlap each other. Because the cosine graph is only a phase shift of the sine graph, one or the other can be used to model motion, but both are not necessary. Most often sine is used.

Amplitude

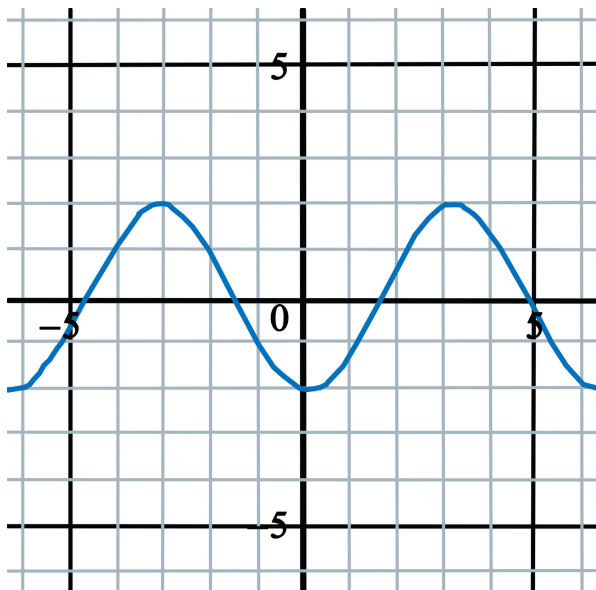
The amplitude and period of a sinusoidal function represent the height and cycle length of a **curve**, respectively, which are important characteristics of the waveform. Amplitude and period are important in the study of music and acoustics; amplitude indicates the energy, or loudness, of a sound and the period determines its pitch (pitch, or frequency, is the inverse of the period). Amplitude is the height from the mean, or rest, value of the function to its maximum or minimum. The period is the distance required for the function to complete one full cycle. For the sine function

$f(x) = A\sin(Bx)$, the amplitude is given by $|A|$ and the period is defined as $\frac{2\pi}{|B|}$. The same is true for a cosine function.

This function has an amplitude of 1 because the graph goes one unit up and one unit down from the midline of the graph. This function has a period of 2π because the sine wave repeats every 2π units. The amplitude of a periodic function is half the difference between the maximum and minimum values. It describes the height of the graph both above and below a horizontal line passing through the “middle” of the graph.



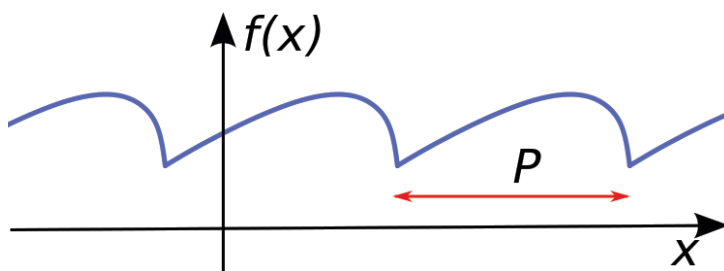
The amplitude is the distance your function reaches above the midline. The amplitude really only applies to the sine and cosine functions.



Without an A value, the amplitude for your sine and cosine functions is 1. You can see this on the graphs of these functions since the graph goes up to 1 at its highest point. If you change your A , then you will also be changing your amplitude. An A of 2 gives you an amplitude of 2. For example $2 \cos(x)$ has an amplitude of 2.

Period

A periodic function is a function that repeats its values in regular intervals or periods. The most important examples are the trigonometric functions, which repeat over intervals of 2π radians. Periodic functions are used throughout science to describe oscillations, waves, and other phenomena that exhibit periodicity. Any function that is not periodic is called aperiodic.



A function f is said to be **periodic** if, for some **nonzero** constant P , we have

$$f(x + P) = f(x)$$

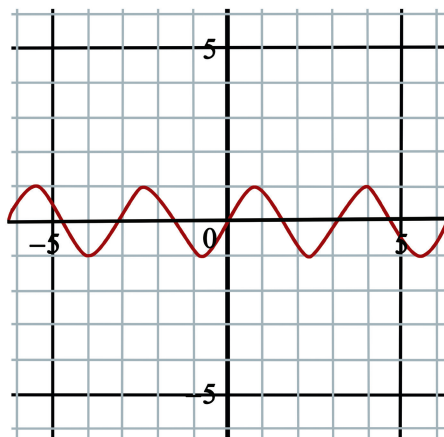
for all values of x in the domain. A nonzero constant P for which this is the case is called a period of the function. If there exists a least positive constant P with this property, it is called the fundamental period (also primitive period, basic period, or prime period.) Often, “the” period of a function is used to mean its fundamental period. A function with period P will repeat on intervals of length P , and these intervals are sometimes also referred to as periods of the function.

Geometrically, a periodic function can be defined as a function whose graph exhibits translational symmetry. Specifically, a function f is periodic with period P if the graph of f is invariant under translation in the x -direction by a distance of P . This definition of periodic can be extended to other geometric shapes and patterns, as well as be generalized to higher dimensions, such as periodic tessellations of the plane. A sequence can also be viewed as a function defined on the natural numbers, and for a periodic sequence these notions are defined accordingly. The period of your graph is how often the graph repeats itself. This is found by dividing your regular period by the absolute value of B .

For sine and cosine functions, the regular period is 2π . This means your graph will repeat itself every 2π , so at 6.28, at 12.56, etc. If your function has a B value, then it will change that period. For example, say you have this function:

$$f(x) = \sin(2x)$$

Because you now have a value of 2 for B , your period is no longer 2π . It now is $2\pi / 2 = \pi$. So now your graph will repeat every π , so at 3.14, 6.28, 9.42, etc.



What do you think your period will be if your B value is 3? If you said $2\pi / 3$, then you are 100% right!

For tangent and cotangent functions, the regular period is π . So if your B value here is 2, then your new period is $\pi / 2$ or roughly 1.57.

Examples

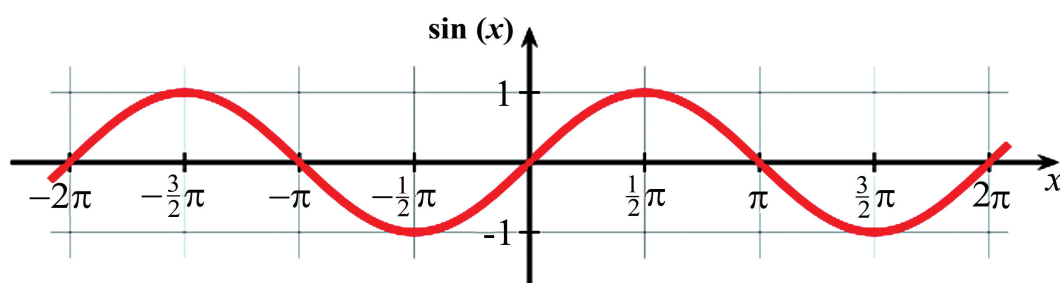
For example, the sine function is periodic with period 2π , since

$$\sin(x + 2\pi) = \sin x$$

for all values of x . This function repeats on intervals of length 2π (see the graph to the right)

Everyday examples are seen when the variable is *time*; for instance the hands of a clock or the phases of the moon show periodic behavior. **Periodic motion** is motion in which the position(s) of the system are expressible as periodic functions, all with the *same* period.

For a function on the real numbers or on the integers, that means that the entire graph can be formed from copies of one particular portion, repeated at regular intervals.



A simple example of a periodic function is the function f that gives the “fractional part” of its argument. Its period is 1. In particular,

$$f(0.5) = f(1.5) = f(2.5) = \cdots = 0.5$$

The graph of the function f is the sawtooth wave.

The trigonometric functions sine and cosine are common periodic functions, with period 2π . The subject of Fourier series investigates the idea that an ‘arbitrary’ periodic function is a sum of trigonometric functions with matching periods.

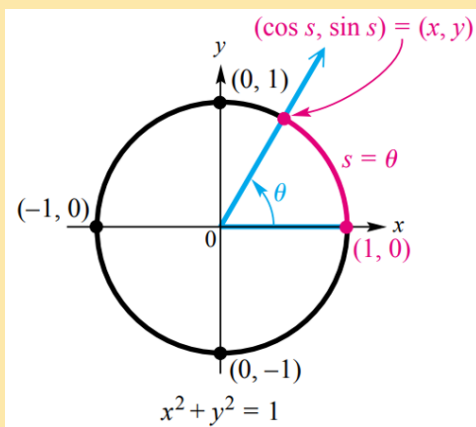
CONCEPT OF FINDING VALUES OF CIRCULAR FUNCTIONS

The circular functions (functions of real numbers) are closely related to the trigonometric functions of angles measured in radians. To see this, let us assume that angle θ is in standard position, superimposed on the unit circle, as shown in Figure. Suppose further that θ is the radian measure of this angle. Using the arc length formula $s=r\theta$ with $r=1$ we have $s=\theta$. Thus, the length of the intercepted arc is the real number that corresponds to the radian measure of θ . Using the definitions of the trigonometric functions, we have

$$\sin \theta = \frac{y}{r} = \frac{y}{1} = y = \sin s,$$

and

$$\cos \theta = \frac{x}{r} = \frac{x}{1} = x = \cos s,$$



and so on. As shown here, the trigonometric functions and the circular functions lead to the same function values, provided we think of the angles as being in radian measure.

SUMMARY

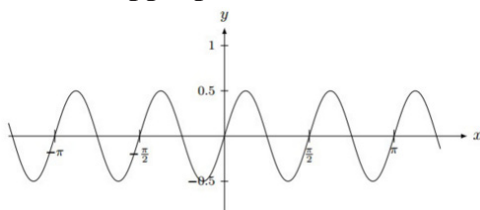
- The circular functions are a group of trigonometric functions that relate angles to the coordinates of points on the unit circle.
- When graphed, the sine and cosine functions create sinusoidal curves that repeat every 360 degrees or 2π radians.
- The tangent function ($\tan$) is defined as the ratio of sine to cosine and can be graphed as well. However, its graph has vertical asymptotes where the cosine is zero (at odd multiples of $\pi/2$ radians or 90-degree intervals), resulting in an oscillating curve with rapid changes.
- The $\sin$ and $\cos$ (and $\csc$ and $\sec$) functions start repeating after 2π radians, and it turns out that the $\tan$ and $\cot$ functions start repeating again after only π radians.
- A complete repetition of the pattern of the function is called a cycle and the period is the horizontal length of one complete cycle.
- Graphing a trigonometric function is actually pretty easy if you know what numbers to look at. Recall that a trigonometric function ('trig function') is a mathematical function of an angle.
- The function $\tan x$ has a very different kind of graph to those for the sine and cosine functions. The period of $\tan x$ is π , rather than 2π , and the amplitude is not defined.
- The amplitude and period of a sinusoidal function represent the height and cycle length of a curve, respectively, which are important characteristics of the waveform.
- Amplitude and period are important in the study of music and acoustics; amplitude indicates the energy, or loudness, of a sound and the period determines its pitch (pitch, or frequency, is the inverse of the period).
- A periodic function is a function that repeats its values in regular intervals or periods. The most important examples are the trigonometric functions, which repeat over intervals of 2π radians.

MULTIPLE CHOICE QUESTIONS

1. The period of $3\sec x/3$ is

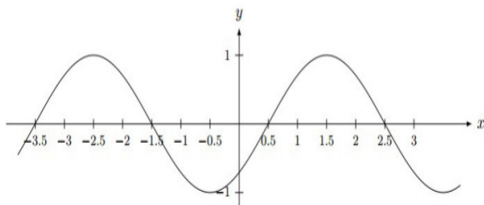
- a. π
- b. 2π
- c. 3π
- d. 6π

2. Find an appropriate function for the following graph:



- a. $y = 0.5 \cos(4\pi x)$
- b. $y = 0.5 \sin(x/4)$
- c. $y = 0.5 \sin(4x)$
- d. $y = 0.5 \cos(4x)$

3. Which of the functions below correspond to the given graph?



- a. $y = \sin(x/2 - 1/4)$
- b. $y = \sin(\pi x/2 - \pi/4)$
- c. $y = \cos(\pi x/2 - \pi/4)$
- d. $y = \sin(\pi x/2)$

4. The period of $\tan(x)$ is

- a. $\pi/3$
- b. $\pi/2$
- c. $2\pi/3$
- d. 2π

5. The period of $15\csc(x/3)$ is

- a. 15π
- b. 10π
- c. 5π
- d. 2π

Answer

1. (d) 2. (c) 3. (b) 4. (d) 5. (b)

REVIEW QUESTIONS

1. Discuss about graphing trigonometric functions.
2. Explain the term with example amplitude and phase shift.
3. Is the graph of $y = \sin x$ symmetric with respect to a reflection in the origin?
4. Is the graph of $y = \sin x$ symmetric with respect to a translation $T_{-2\pi, 0}$?
5. What is the maximum and minimum value of y on the graph of $y = \cos x$?
6. What is the period of the cosine function?

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INVERSE TRIGONOMETRIC FUNCTIONS



The inverse trigonometric functions are the inverse functions of the trigonometric functions (with suitably restricted domains).

After studying this chapter, you will understand the core concepts of:

- Inverse Function
- Trigonometric Equations

INTRODUCTION

Inverse trigonometric functions are defined as the inverse functions of the basic trigonometric functions, which are sine, cosine, tangent, cotangent, secant and cosecant functions. They are also termed arcus functions, antitrigonometric functions or cyclometric functions. These inverse functions in trigonometry are used to get the angle with any of the trigonometry ratios. The inverse trigonometry functions have major applications in the field of engineering, physics, geometry and navigation.

The inverse trigonometric functions are essential for solving trigonometric equations and for dealing with various problems involving angles and triangles. They are often used in calculus, geometry, physics, and engineering.

Inverse Trig Functions

$\sin^{-1}(-\frac{1}{2}) = ?$	$\sin^{-1}(-1) = ?$
$\cos^{-1}(0) = ?$	$\cos^{-1}(-\frac{\sqrt{3}}{2}) = ?$
$\tan^{-1}(\sqrt{3}) = ?$	$\tan^{-1}(-\frac{\sqrt{3}}{3}) = ?$

Inverse trigonometric functions are also called “arc functions” since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value. The inverse trigonometric functions perform the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant and cotangent. We know that trigonometric functions are applicable, especially to the right-angle triangle. These six important functions are used to find the angle measure in the right-angle triangle when two sides of the triangle measures are known.

$$\begin{aligned}
 y &= 3x^2 + 6x + 9 \\
 \Rightarrow y &= 3\{x - (-1)\}^2 + 6 \\
 \Rightarrow y - 6 &= 3\{x - (-1)\}^2 \\
 \Rightarrow \frac{y-6}{3} &= \frac{3\{x - (-1)\}^2}{3}
 \end{aligned}$$

CORE CONCEPT: INVERSE FUNCTION

Every mathematical function, from the simplest to the most complex, has an inverse. In mathematics, inverse usually means opposite. For addition, the inverse is subtraction. For multiplication, it's division. And for trigonometric functions, it's the inverse trigonometric

functions. Trigonometric functions are the functions of an angle. The term function is used to describe the relationship between two sets of numbers or variables. In modern mathematics, there are six basic trigonometric functions: sine, cosine, tangent, secant, cosecant and cotangent. The inverse of these functions are inverse sine, inverse cosine, inverse tangent, inverse secant, inverse cosecant and inverse cotangent.

Inverse of a function 'f' exists, if the function is one-one and onto, i.e, bijective. Since trigonometric functions are many-one over their domains, we restrict their domains and co-domains in order to make them one-one and onto and then find their inverse. The domains and ranges (principal value branches) of inverse trigonometric functions are given below:

Functions	Domain	Range (Principal value branches)
$y = \sin^{-1}x$	$[-1, 1]$	$\frac{-\pi}{2}, \frac{\pi}{2}$
$y = \cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$y = \operatorname{cosec}^{-1}x$	$\mathbf{R} - (-1, 1)$	$\frac{-\pi}{2}, \frac{\pi}{2} - \{0\}$
$y = \sec^{-1}x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \frac{\pi}{2}$
$y = \tan^{-1}x$	$\mathbf{R}$	$\frac{-\pi}{2}, \frac{\pi}{2}$
$y = \cot^{-1}x$	$\mathbf{R}$	$(0, \pi)$

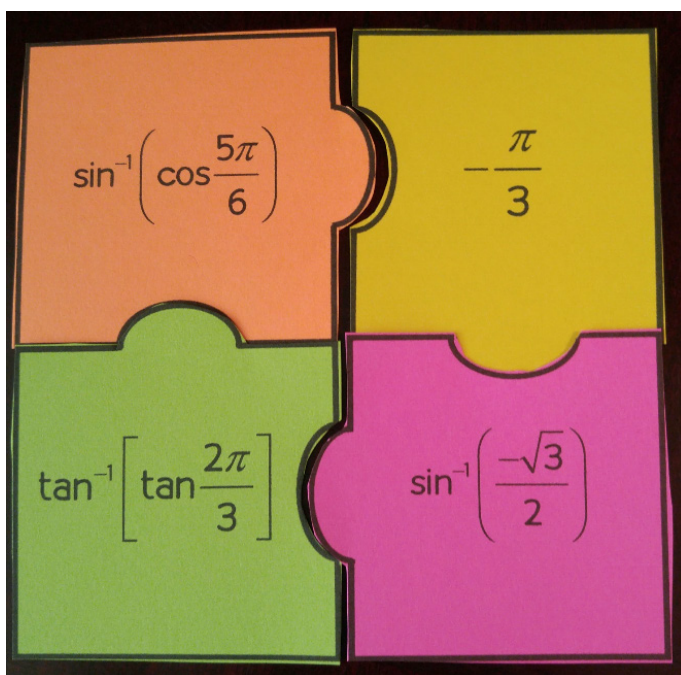
Notes:

- The symbol $\sin^{-1}x$ should not be confused with $(\sin x)^{-1}$. Infact $\sin^{-1}x$ is an angle, the value of whose sine is x, similarly for other trigonometric functions.
- The smallest numerical value, either positive or negative, of θ is called the principal value of the function.
- Whenever no branch of an inverse trigonometric function is mentioned, we mean the principal value branch. The value of the inverse

trigonometric function which lies in the range of principal branch is its principal value.

Graph of an Inverse Trigonometric Function

The graph of an inverse trigonometric function can be obtained from the graph of original function by interchanging x-axis and y-axis, i.e, if (a, b) is a point on the graph of trigonometric function, then (b, a) becomes the corresponding point on the graph of its inverse trigonometric function. It can be shown that the graph of an inverse function can be obtained from the corresponding graph of original function as a mirror image (i.e., reflection) along the line $y = x$.



Properties of Inverse Trigonometric Functions

1.

$$\sin^{-1}(\sin x) = x \quad : \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos^{-1}(\cos x) = x \quad : \quad x \in [0, \pi]$$

$$\tan^{-1}(\tan x) = x \quad : \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\cot^{-1}(\cot x) = x \quad : \quad x \in (0, \pi)$$

$$\sec^{-1}(\sec x) = x \quad : \quad x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x \quad : \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$$

2.

$$\begin{array}{ll}
 \sin(\sin^{-1}x) = x & : \quad x \in [-1, 1] \\
 \cos(\cos^{-1}x) = x & : \quad x \in [-1, 1] \\
 \tan(\tan^{-1}x) = x & : \quad x \in \mathbf{R} \\
 \cot(\cot^{-1}x) = x & : \quad x \in \mathbf{R} \\
 \sec(\sec^{-1}x) = x & : \quad x \in \mathbf{R} - (-1, 1) \\
 \operatorname{cosec}(\operatorname{cosec}^{-1}x) = x & : \quad x \in \mathbf{R} - (-1, 1)
 \end{array}$$

3.

$$\sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1}x : \quad x \in \mathbf{R} - (-1, 1)$$

$$\cos^{-1} \frac{1}{x} = \sec^{-1}x : \quad x \in \mathbf{R} - (-1, 1)$$

$$\tan^{-1} \frac{1}{x} = \cot^{-1}x : \quad x > 0$$

$$= -\pi + \cot^{-1}x : \quad x < 0$$

4.

$$\sin^{-1}(-x) = -\sin^{-1}x : \quad x \in [-1, 1]$$

$$\cos^{-1}(-x) = \pi - \cos^{-1}x : \quad x \in [-1, 1]$$

$$\tan^{-1}(-x) = -\tan^{-1}x : \quad x \in \mathbf{R}$$

$$\cot^{-1}(-x) = \pi - \cot^{-1}x : \quad x \in \mathbf{R}$$

$$\sec^{-1}(-x) = \pi - \sec^{-1}x : \quad x \in \mathbf{R} - (-1, 1)$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x : \quad x \in \mathbf{R} - (-1, 1)$$

5.

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} : \quad x \in [-1, 1]$$

$$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} : \quad x \in \mathbf{R}$$

$$\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2} : \quad x \in \mathbf{R} - [-1, 1]$$

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy} : \quad xy < 1$$

6.

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x+y}{1-xy} : \quad xy < 1$$

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) : \quad xy > -1$$

7.

$$2\tan^{-1}x = \sin^{-1} \frac{2x}{1-x^2} \quad : \quad -1 \leq x \leq 1$$

$$2\tan^{-1}x = \cos^{-1} \frac{1-x^2}{1+x^2} \quad : \quad x \geq 0$$

$$2\tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2} \quad : \quad -1 < x < 1$$

Solved Examples

Example 1

Find the principal value of $\cos^{-1}x$, for $x = \frac{\sqrt{3}}{2}$

Solution

$$\text{If } \cos^{-1} \frac{\sqrt{3}}{2} = \theta, \text{ then } \cos \theta = \frac{\sqrt{3}}{2}.$$

Since we are considering principal branch, $\theta \in [0, \pi]$. Also, since $\frac{\sqrt{3}}{2} > 0$, θ being in the first quadrant, hence $\cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

Example 2

Evaluate $\tan^{-1} \sin \frac{-\pi}{2}$

Solution

$$\tan^{-1} \sin \frac{-\pi}{2} = \tan^{-1} \left(-\sin \left(\frac{\pi}{2} \right) \right) = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

Example 3

Find the value of $\cos^{-1} \cos \frac{13\pi}{6}$

Solution

$$\begin{aligned} \cos^{-1} \cos \frac{13\pi}{6} &= \cos^{-1} \left(\cos \left(2\pi + \frac{\pi}{6} \right) \right) = \cos^{-1} \left(\cos \frac{\pi}{6} \right) \\ &= \frac{\pi}{6}. \end{aligned}$$

Example 4

Find the value of $\tan^{-1} \tan \frac{9\pi}{8}$

Solution

$$\tan^{-1} \tan \frac{9\pi}{8} = \tan^{-1} \tan \left(\pi + \frac{\pi}{8} \right)$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{8} \right) \right) = \frac{\pi}{8}$$

Example 5

Evaluate $\tan (\tan^{-1}(-4))$.

Solution

Since $\tan (\tan^{-1} x) = x, \forall x \in \mathbb{R}, \tan (\tan^{-1}(-4)) = -4$.

Example 6

Evaluate: $\tan^{-1} 3 - \sec^{-1} (-2)$.

Solution

$$\tan^{-1} \sqrt{3} - \sec^{-1} (-2) = \tan^{-1} \sqrt{3} - [\pi - \sec^{-1} 2]$$

$$= \frac{\pi}{3} - \pi + \cos^{-1} \left(\frac{1}{2} \right) = -\frac{2\pi}{3} + \frac{\pi}{3} = -\frac{\pi}{3}$$

Example 7

$$\text{Evaluate: } \sin^{-1} \cos \sin^{-1} \frac{\sqrt{3}}{2}$$

Solution

$$\sin^{-1} \cos \sin^{-1} \frac{\sqrt{3}}{2} = \sin^{-1} \cos \frac{\pi}{3} = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$$

Example 8

Prove that $\tan (\cot^{-1} x) = \cot (\tan^{-1} x)$. State with reason whether the **equality** is valid for all values of x .

Solution

Let $\cot^{-1} x = \theta$. Then $\cot \theta = x$

$$\text{or, } \tan \frac{\pi}{2} - \theta = x \Rightarrow \tan^{-1} x = \frac{\pi}{2} - \theta$$

$$\text{So } \tan(\cot^{-1} x) = \tan \theta = \cot \left(\frac{\pi}{2} - \theta \right) = \cot \left(\frac{\pi}{2} - \cot^{-1} x \right) = \cot(\tan^{-1} x)$$

The equality is valid for all values of x since $\tan^{-1} x$ and $\cot^{-1} x$ are true for $x \in \mathbb{R}$

Example 9

Find the value of $\sec \left(\tan^{-1} \frac{y}{2} \right)$

In mathematics, **equality** is a relationship between two quantities or, more generally two mathematical expressions, asserting that the quantities have the same value, or that the expressions represent the same mathematical object.

Solution

Let $\tan^{-1} \frac{y}{2} = \theta$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, $\tan \theta = \frac{y}{2}$,

which gives

$$\sec \theta = \frac{\sqrt{4 + y^2}}{2}$$

Therefore,

$$\sec\left(\tan^{-1} \frac{y}{2}\right) = \sec \theta = \frac{\sqrt{4 + y^2}}{2}$$

Example 10

Find value of $\tan(\cos^{-1} x)$ and hence evaluate $\tan^{-1} \frac{8}{17}$

Solution

Let $\cos^{-1} x = \theta$, then $\cos \theta = x$, where $\theta \in [0, \pi]$

$$\text{Therefore, } \tan(\cos^{-1} x) = \tan \theta = \frac{\sqrt{1 - \cos^2 \theta}}{\cos \theta} = \frac{\sqrt{1 - x^2}}{x}$$

$$\text{Hence } \tan\left(\cos^{-1} \frac{8}{17}\right) = \frac{\sqrt{1 - \left(\frac{8}{17}\right)^2}}{\frac{8}{17}} = \frac{15}{8}$$

Example 11

Find the value of $\sin^{-1} 2 \cot^{-1} \frac{-5}{12}$

Solution

$$\text{Let } \cot^{-1} \left(\frac{-5}{12}\right) = y. \quad \text{Then } \cot y = \frac{-5}{12}$$

$$\text{Now } \sin^{-1} 2 \cot^{-1} \frac{-5}{12} = \sin^{-1} 2y$$

$$= 2 \sin y \cos y = 2 \cdot \frac{12}{13} \cdot \frac{-5}{13} \quad \left[\text{since } \cot y < 0, \text{ so } y \in \left(\frac{\pi}{2}, \pi\right) \right]$$

$$= \frac{-120}{169}$$

Example 12

$$\text{Evaluate } \cos^{-1} \frac{1}{4} - \sec^{-1} \frac{4}{3}$$

$$\cos^{-1} \frac{1}{4} - \sec^{-1} \frac{4}{3} = \cos^{-1} \left[\sin^{-1} \frac{1}{4} + \cos^{-1} \frac{3}{4} \right]$$

$$\begin{aligned}
&= \cos \sin^{-1} \frac{1}{4} \cos \cos^{-1} \frac{3}{4} - \sin \sin^{-1} \frac{1}{4} \sin \cos^{-1} \frac{3}{4} \\
&= \frac{3}{4} \sqrt{1 - \frac{1}{4}^2} - \frac{1}{4} \sqrt{1 - \frac{3}{4}^2} \\
&= \frac{3}{4} \frac{\sqrt{15}}{4} - \frac{1}{4} \frac{\sqrt{7}}{4} = \frac{3\sqrt{15} - \sqrt{7}}{16}
\end{aligned}$$

Long Answer (L.A.)

Example 13

Prove that $2\sin^{-1} \frac{3}{5} - \tan^{-1} \frac{17}{31} = \frac{\pi}{4}$

Solution

Let $\sin^{-1} \frac{3}{5} = \theta$, then $\sin \theta = \frac{3}{5}$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Thus

$$\tan \theta = \frac{3}{4}, \text{ which gives } \theta = \tan^{-1} \frac{3}{4}.$$

Therefore,

$$\begin{aligned}
&2\sin^{-1} \frac{3}{5} - \tan^{-1} \frac{17}{31} \\
&= 2\theta - \tan^{-1} \frac{17}{31} = 2 \tan^{-1} \frac{3}{4} - \tan^{-1} \frac{17}{31} \\
&= \tan^{-1} \left(\frac{2 \cdot \frac{3}{4}}{1 - \frac{9}{16}} \right) - \tan^{-1} \frac{17}{31} = \tan^{-1} \frac{24}{7} - \tan^{-1} \frac{17}{31} \\
&= \tan^{-1} \left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \cdot \frac{17}{31}} \right) = \frac{\pi}{4}
\end{aligned}$$

Example 14

Prove that

$$\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$$

Solution

We have

$$\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$$

$$= \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} \quad (\text{since } \cot^{-1} x = \tan^{-1} \frac{1}{x}, \text{ if } x > 0)$$

$$= \tan^{-1} \left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}} \right) + \tan^{-1} \frac{1}{18} \quad (\text{since } x \cdot y = \frac{1}{7} \cdot \frac{1}{8} < 1)$$

$$= \tan^{-1} \frac{3}{11} + \tan^{-1} \frac{1}{18} = \tan^{-1} \left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}} \right) \quad (\text{since } xy < 1)$$

$$= \tan^{-1} \frac{65}{195} = \tan^{-1} \frac{1}{3} = \cot^{-1} 3$$

Example 15

Which is greater, $\tan 1$ or $\tan^{-1} 1$?

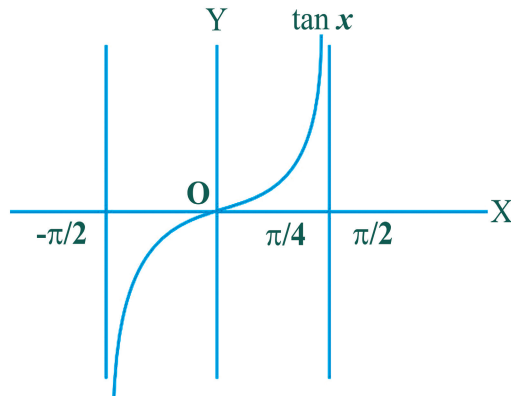
Solution From Fig. 1, we note that $\tan x$ is an increasing function in the interval

$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, since $1 > \frac{\pi}{4} \Rightarrow \tan 1 > \tan \frac{\pi}{4}$. This gives

$$\tan 1 > 1$$

$$\Rightarrow \tan 1 > 1 > \frac{\pi}{4}$$

$$\Rightarrow \tan 1 > 1 > \tan^{-1}(1).$$



Example 16

Find the value of

$$\sin \left(2 \tan^{-1} \frac{2}{3} \right) + \cos (\tan^{-1} \sqrt{3}).$$

Solution

Let $\tan^{-1} \frac{2}{3} = x$ and $\tan^{-1} \sqrt{3} = y$ so that $\tan x = \frac{2}{3}$ and $\tan y = \sqrt{3}$.

Therefore,

$$\begin{aligned}
 & \sin\left(2 \tan^{-1} \frac{2}{3}\right) + \cos(\tan^{-1} \sqrt{3}) \\
 &= \sin(2x) + \cos y \\
 &= \frac{2 \tan x}{1 + \tan^2 x} + \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{2 \cdot \frac{2}{3}}{1 + \frac{4}{9}} + \frac{1}{1 + \sqrt{(\sqrt{3})^2}} \\
 &= \frac{12}{13} + \frac{1}{2} = \frac{37}{26}
 \end{aligned}$$

Example 17

Solve for x

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x, \quad x > 0$$

Solution

$$\text{From given equation, we have } 2 \tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1} x$$

$$\Rightarrow 2 \left[\tan^{-1} 1 - \tan^{-1} x \right] = \tan^{-1} x$$

$$\Rightarrow 2 \left(\frac{\pi}{4} \right) = 3 \tan^{-1} x \Rightarrow \frac{\pi}{6} = \tan^{-1} x$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

Example 18

Find the values of x which satisfy the equation

$$\sin^{-1} x + \sin^{-1} (1-x) = \cos^{-1} x.$$

Solution

From the given equation, we have

$$\begin{aligned}
 & \sin(\sin^{-1} x + \sin^{-1} (1-x)) = \sin(\cos^{-1} x) \\
 \Rightarrow & \sin(\sin^{-1} x) \cos(\sin^{-1} (1-x)) + \cos(\sin^{-1} x) \sin(\sin^{-1} (1-x)) = \sin(\cos^{-1} x) \\
 \Rightarrow & x \sqrt{1-(1-x)^2} + (1-x) \sqrt{1-x^2} = \sqrt{1-x^2} \\
 \Rightarrow & x \sqrt{2x-x^2} + \sqrt{1-x^2} (1-x-1) = 0 \\
 \Rightarrow & x \left(\sqrt{2x-x^2} - \sqrt{1-x^2} \right) = 0 \\
 \Rightarrow & x = 0 \quad \text{or} \quad 2x - x^2 = 1 - x^2
 \end{aligned}$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = \frac{1}{2}$$

Example 19

Solve the equation $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}$

Solution

From the given equation, we have $\sin^{-1} 6x = -\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}x$

$$\Rightarrow \sin(\sin^{-1} 6x) = \sin\left(-\frac{\pi}{2} - \sin^{-1} 6\sqrt{3}x\right)$$

$$\Rightarrow 6x = -\cos(\sin^{-1} 6\sqrt{3}x)$$

$$\Rightarrow 6x = -\sqrt{1-108x^2} \quad \text{Squaring, we get}$$

$$36x^2 = 1 - 108x^2$$

$$\Rightarrow 144x^2 = 1 \quad \Rightarrow x = \pm \frac{1}{12}$$

Note that $x = -\frac{1}{12}$ is the only root of the equation as $x = 1/12$ does not satisfy it

Example 20

Show that

$$2 \tan^{-1} \left\{ \tan \frac{\alpha}{2} \cdot \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right\} = \tan^{-1} \frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta}$$

Solution

$$\text{L.H.S.} = \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)} \quad \left(\text{since } 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right)$$

$$= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}}}{1 - \tan^2 \frac{\alpha}{2} \cdot \left(\frac{1 - \tan \frac{\beta}{2}}{1 + \tan \frac{\beta}{2}} \right)^2}$$

$$= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \cdot \left(1 - \tan^2 \frac{\beta}{2} \right)}{\left(1 + \tan^2 \frac{\beta}{2} \right)^2 - \tan^2 \frac{\alpha}{2} \left(1 - \tan^2 \frac{\beta}{2} \right)^2}$$

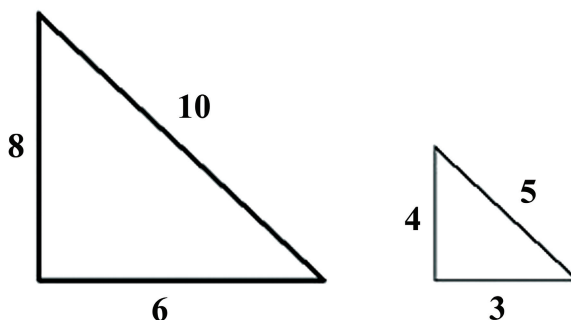
$$= \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \left(1 - \tan^2 \frac{\beta}{2} \right)}{\left(1 + \tan^2 \frac{\beta}{2} \right) \left(1 - \tan^2 \frac{\alpha}{2} \right) + 2 \tan \frac{\beta}{2} \left(1 + \tan^2 \frac{\alpha}{2} \right)}$$

$$\begin{aligned}
 &= \tan^{-1} \frac{\frac{2 \tan \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} \cdot \frac{1 - \tan^2 \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}}{\frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} + \frac{2 \tan \frac{\beta}{2}}{1 + \tan^2 \frac{\beta}{2}}} \\
 &= \tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta} \right) = \text{R.H.S.}
 \end{aligned}$$

The notion that there should be some standard correspondence between the lengths of the sides of a triangle and the angles of the triangle comes as soon as one recognizes that similar triangles maintain the same ratios between their sides.

Trigonometric Ratios

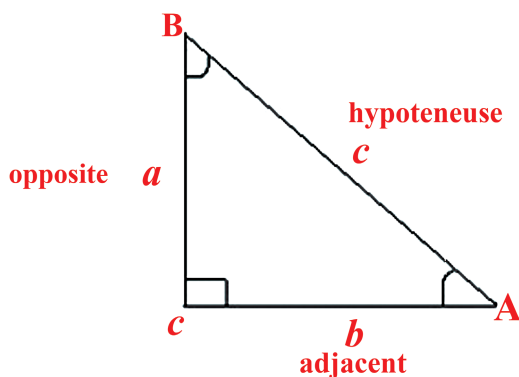
The trigonometric functions can all be defined as ratios of the sides of a right triangle. Since all right triangles conform to the Pythagorean Theorem, as long as the angles of two right triangles are the same, their sides will be proportional. Because of this, the ratios of one side to another will always be the same. Take a look at this example.



These triangles have the same angle measures, so their sides are proportional. Any ratio of one side to another will be the same for both triangles.

$$6/10 = 3/5$$

By discovering that these ratios are the same for any sized right triangle (as long as they have the same angle measure), the trigonometric functions were discovered. These functions relate one angle of a triangle to the ratio of two of its sides.



Name	Ratio	Notation
Sine	opposite/hypotenuse	$\sin(\theta)$
Cosine	adjacent/hypotenuse	$\cos(\theta)$
Tangent	opposite/adjacent	$\tan(\theta)$
Cosecant (1/Sine)	hypotenuse/opposite	$\operatorname{Cosec}(\theta)$ or $\operatorname{csc}(\theta)$
Secant (1/Cosine)	hypotenuse/adjacent	$\sec(\theta)$
Cotangent (1/Tangent)	adjacent/opposite	$\cot(\theta)$

Because of these ratios, when an angle (other than the right angle) of a right triangle and at least one side are known, you can determine the length of the other sides using these ratios. And inversely, when the lengths of two sides are known, the angle measure can be determined.

Equivalent to the right-triangle definitions, the trigonometric functions can also be defined in terms of the rise, run, and slope of a line segment relative to horizontal.

Since memorizing these ratios can prove to be difficult, there is a mnemonic that helps keep them straight. SOH CAH TOA is a helpful device to remember which ratio goes with which function.

Sine = Opposite/Hypotenuse

Cosine = Adjacent/Hypotenuse

Tangent = Opposite/Adjacent

Inverse Trigonometric Functions

Inverse Sine Function (Arcsine) Each of the trigonometric functions sine, cosine, tangent, secant, cosecant and cotangent has an inverse (with a restricted domain).

The inverse trigonometric functions are used to determine the angle measure when at least two sides of a right triangle are known. The particular function that should be used depends on what two sides are known. For example, if you know the hypotenuse and the side opposite the angle in question, you could use the **inverse sine function**. If you know the side opposite and the side adjacent to the angle in question, the inverse tangent is the function you need. There are two methods for determining an inverse trigonometric function. The first is by using a table containing all the results for every ratio. It can be tedious and cumbersome. The other is using a scientific calculator. The inverse functions for the sine, cosine and tangent can be determined quickly.



CORE CONCEPT: TRIGONOMETRIC EQUATIONS

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' which means measuring the sides of a triangle. An angle is the amount of rotation of a revolving line with respect to a fixed line.

If the rotation is in clockwise direction the angle is negative and it is positive if the rotation is in the anti-clockwise direction.

In order to solve these equations we shall make extensive use of the graphs of the functions sine, cosine and tangent. The symmetries which are apparent in these graphs, and their periodicities are particularly important as we shall see.

Some special angles and their trigonometric ratios

In the examples which follow a number of angles and their trigonometric ratios are used frequently. We list these angles and their sines, cosines and tangents.

Table 1:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Some simple trigonometric equations

Example

Suppose we wish to solve the equation $\sin x = 0.5$ and we look for all solutions lying in the interval $0^\circ \leq x \leq 360^\circ$. This means we are looking for all the angles, x , in this interval which have a sine of 0.5. We begin by sketching a graph of the function $\sin x$ over the given interval. This is shown in Figure 1.

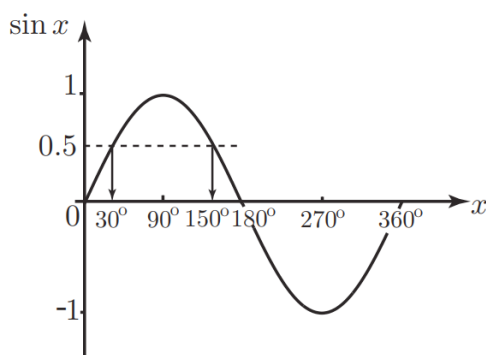


Figure 1: A graph of $\sin x$.

We have drawn a dotted horizontal line on the graph indicating where $\sin x = 0.5$. The solutions of the given equation correspond to the points where this line crosses the curve. From the Table above we note that the first angle with a sine equal to 0.5 is 30° . This is indicated in Figure 1. Using the symmetries of the **graph**, we can deduce all the angles which have a sine of 0.5.

These are:

$$x = 30^\circ, 150^\circ$$

This is because the second solution, 150° , is the same distance to the left of 180° that the first is to the right of 0° . There are no more solutions within the given interval.

Example

Suppose we wish to solve the equation $\cos x = -0.5$ and we look for all solutions lying in the interval $0 \leq x \leq 360^\circ$.

As before we start by looking at the graph of $\cos x$. This is shown in Figure 2. We have drawn a dotted horizontal line where $\cos x = -0.5$. The solutions of the equation correspond to the points where this line intersects the curve. One fact we do know from the Table 1 is that $\cos 60^\circ = +0.5$. This is indicated on the graph. We can then make use of the symmetry to deduce that the first angle with a cosine equal to -0.5 is 120° . This is because the angle must be the same distance to the right of 90° that 60° is to the left. From the graph we see, from consideration of the symmetry, that the remaining solution we seek is 240° . Thus

$$x = 120^\circ, 240^\circ$$

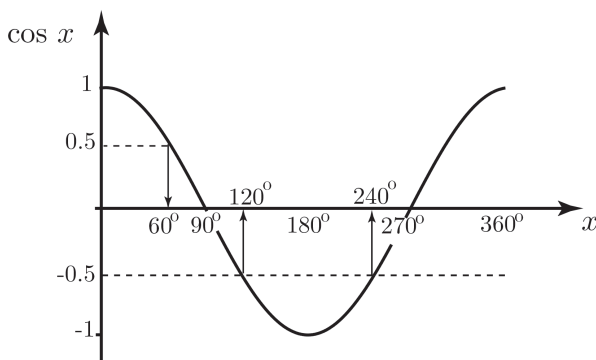


Figure 2: A graph of $\cos x$.

Example

Suppose we wish to solve $\sin 2x = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 360^\circ$.

Note that in this case we have the sine of a multiple angle, $2x$.

To enable us to cope with the multiple angle we shall consider a new variable u where $u = 2x$, so the problem becomes that of solving

$$\sin u = \frac{\sqrt{3}}{2} \quad \text{for } 0 \leq u \leq 720^\circ$$

We draw a graph of $\sin u$ over this interval as shown in Figure 3.

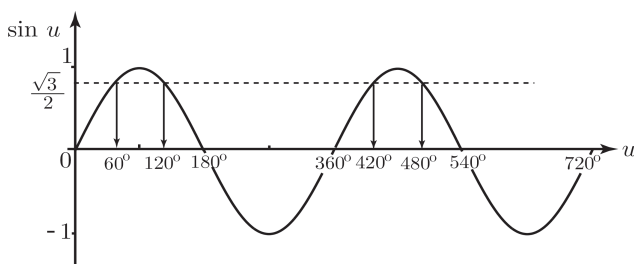


Figure 3: A graph of $\sin u$ for u lying between 0 and 720° .

By referring to the Table 1, we know that $\sin 60^\circ = \frac{\sqrt{3}}{2}$. This is indicated on the graph. From the graph we can deduce another angle which has a sine of $\frac{\sqrt{3}}{2}$. This is 120° . Because of the periodicity we can see there are two more angles, 420° and 480° .

We therefore know all the angles in the interval with sine equal to $\frac{\sqrt{3}}{2}$, namely

$$u = 60^\circ, 120^\circ, 420^\circ, 480^\circ$$

But $u = 2x$ so that

$$2x = 60^\circ, 120^\circ, 420^\circ, 480^\circ$$

from which

$$x = 30^\circ, 60^\circ, 210^\circ, 240^\circ$$

Example

Suppose we wish to solve $\tan 3x = -1$ for values of x in the interval $0^\circ \leq x \leq 180^\circ$. Note that in this example we have the tangent of a multiple angle, $3x$

To enable us to cope with the multiple angle we shall consider a new variable u where $u = 3x$, so the problem becomes that of solving

$$\tan u = -1 \quad \text{for } 0 \leq u \leq 540^\circ$$

We draw a graph of $\tan u$ over this interval as shown in Figure 4.

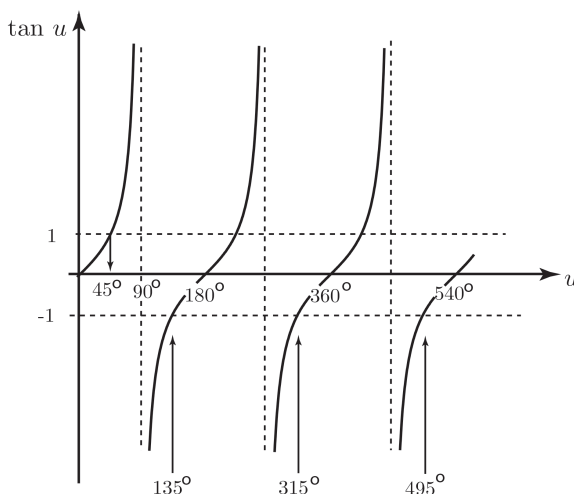


Figure 4: A graph of $\tan u$.

We know from the Table 1, that an angle whose tangent is 1 is 45° , so using the symmetry in the graph we can find the angles which have a tangent equal to -1 . The first will be the same distance to the right of 90° that 45° is to the left that is 135° . The other angles will each be 180° further to the right because of the periodicity of the tangent function. Consequently the solutions of $\tan u = -1$ are given by

$$u = 135^\circ, 315^\circ, 495^\circ,$$

But $u = 3x$ and so

$$3x = 135^\circ, 315^\circ, 495^\circ,$$

from which

$$x = 45^\circ, 105^\circ, 165^\circ$$

Example

Suppose we wish to solve $\cos \frac{x}{2} = -\frac{1}{2}$ for values of x in the interval $0 \leq x \leq 360^\circ$. In this Example we are dealing with the cosine of a multiple angle $\frac{x}{2}$.

To enable us to handle this we make a substitution $u = \frac{x}{2}$ so that the equation becomes $\cos u = -\frac{1}{2}$ for $0 \leq u \leq 180^\circ$. A graph of $\cos u$ over this interval is shown in Figure 5.

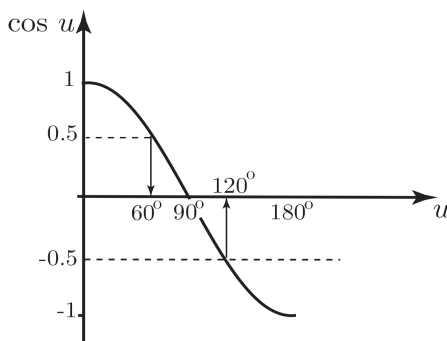


Figure 5: A graph of $\cos u$

We know that the angle whose cosine is $\frac{1}{2}$ is 60° . Using the symmetry in the graph we can find all the angles with a cosine equal to $-\frac{1}{2}$.

But $u = \frac{x}{2}$ and so $x = 2u$. We conclude that there is a single solution, $x = 240^\circ$.

Let us now look at some examples over the interval $-180^\circ \leq x \leq 180^\circ$.

Example

Suppose we wish to solve $\sin x = 1$ for $-180^\circ \leq x \leq 180^\circ$

From the graph of $\sin x$ over this interval, shown in Figure 6, we see there is only one angle which has a sine equal to 1, that is $x = 90^\circ$.

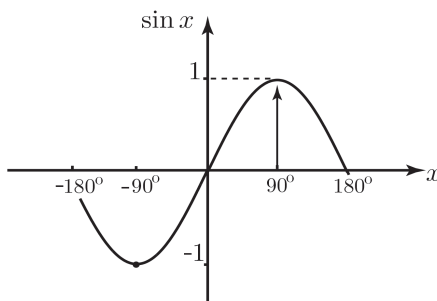


Figure 6: A graph of the sine function.

Example

Suppose we wish to solve $\cos 2x = \frac{1}{2}$ for $-180^\circ \leq x \leq 180^\circ$.

In this Example we have a multiple angle, $2x$.

To handle this we let $u = 2x$ and instead solve

$$\cos u = \frac{1}{2} \quad \text{for } -360^\circ \leq x \leq 360^\circ$$

A graph of the cosine function over this interval is shown in Figure 7.

In mathematics, “**symmetry**” has a more precise definition, that an object is invariant to any of various transformations; including reflection, rotation or scaling.

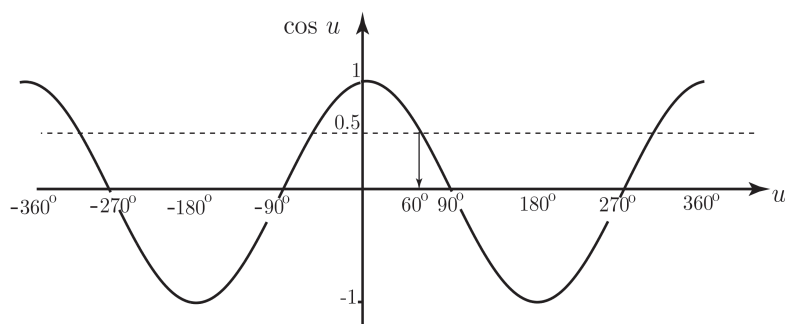


Figure 7: A graph of $\cos u$.

The dotted line indicates where the cosine is equal to $\frac{1}{2}$. Remember we already know one angle which has cosine equal to $\frac{1}{2}$ and this is 60° . From the graph, and making use of **symmetry**, we can deduce all the other angles with cosine equal to $\frac{1}{2}$. These are

$$u = -300^\circ, -60^\circ, 60^\circ, 300^\circ$$

Then $u = 2x$ so that

$$2x = -300^\circ, -60^\circ, 60^\circ, 300^\circ$$

from which

$$x = -150^\circ, -30^\circ, 30^\circ, 150^\circ$$

Example

Suppose we wish to solve $\tan 2x = \sqrt{3}$ for $-180^\circ \leq x \leq 180^\circ$.

We again have a multiple angle, $2x$. We handle this by letting $u = 2x$ so that the problem becomes that of solving

$$\tan u = \sqrt{3} \quad \text{for } -360^\circ \leq u \leq 360^\circ$$

We plot a graph of $\tan u$ between -360° and 360° as shown in Figure 8.

We know from the Table 1, that one angle which has a tangent equal to $\sqrt{3}$ is $u = 60^\circ$. We can use the symmetry of the graph to deduce others. These are

$$u = -300^\circ, -120^\circ, 60^\circ, 240^\circ$$

But $u = 2x$ and so

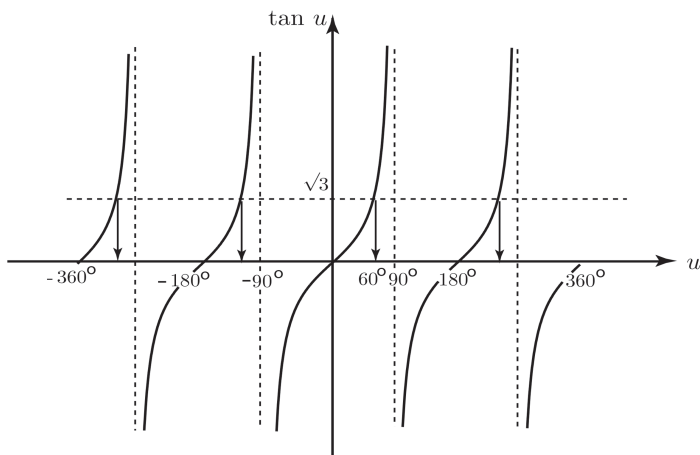


Figure 8: A graph of $\tan u$.

$$2x = -300^\circ, -120^\circ, 60^\circ, 240^\circ$$

and so the required solutions are

$$x = -150^\circ, -60^\circ, 30^\circ, 120^\circ$$

Exercise 1

- Find all the solutions of each of the following equations in the given range

(a) $\sin x = \frac{1}{\sqrt{2}}$ for $0 < x < 360^\circ$

(b) $\cos x = -\frac{1}{\sqrt{2}}$ for $0 < x < 360^\circ$

(c) $\tan x = \frac{1}{\sqrt{3}}$ for $0 < x < 360^\circ$

(d) $\cos x = -1$ for $0 < x < 360^\circ$

- Find all the solutions of each of the following equations in the given range

(a) $\tan x = \sqrt{3}$ for $-180^\circ < x < 180^\circ$

(b) $\tan x = -\sqrt{3}$ for $-180^\circ < x < 180^\circ$

(c) $\cos x = \frac{1}{2}$ for $-180^\circ < x < 180^\circ$

(d) $\sin x = -\frac{1}{\sqrt{2}}$ for $-180^\circ < x < 180^\circ$

- Find all the solutions of each of the following equations in the given range

$$(a) \cos 2x = \frac{1}{\sqrt{2}} \text{ for } -180^\circ < x < 180^\circ$$

$$(b) \tan 3x = 1 \text{ for } -90^\circ < x < 90^\circ$$

$$(c) \sin 2x = \frac{1}{2} \text{ for } -180^\circ < x < 0$$

$$(d) \cos \frac{1}{2}x = -\frac{\sqrt{3}}{2} \text{ for } -180^\circ < x < 180^\circ$$

4. Using identities in the solution of equations

There are many trigonometric identities. Two commonly occurring ones are

$$\sin^2 x + \cos^2 x = 1 \qquad \sec^2 x = 1 + \tan^2 x$$

We will now use these in the solution of trigonometric equations. (If necessary you should refer to the unit entitled Trigonometric Identities).

Example

Suppose we wish to solve the equation $\cos^2 x + \cos x = \sin^2 x$ for $0^\circ \leq x \leq 180^\circ$.

We can use the identity $\sin^2 x + \cos^2 x = 1$, rewriting it as $\sin^2 x = 1 - \cos^2 x$ to write the given equation entirely in terms of cosines.

$$\cos^2 x + \cos x = \sin^2 x$$

$$\cos^2 x + \cos x = 1 - \cos^2 x$$

Rearranging, we can write

$$2 \cos^2 x + \cos x - 1 = 0$$

This is a quadratic equation in which the variable is $\cos x$. This can be factorised to

$$(2 \cos x - 1)(\cos x + 1) = 0$$

Hence

$$2 \cos x - 1 = 0 \qquad \text{or} \qquad \cos x + 1 = 0$$

from which

$$\cos x = \frac{1}{2} \qquad \text{or} \qquad \cos x = -1$$

We solve each of these equations in turn. By referring to the graph of $\cos x$ over the interval $0 \leq x \leq 180^\circ$ which is shown in Figure 9, we see that there is only one solution of the equation $\cos x = \frac{1}{2}$ in this interval, and this is $x = 60^\circ$. From the same graph we can deduce the solution of $\cos x = -1$ to be $x = 180^\circ$.

So there are two solutions of the original equation, 60° and 180° .

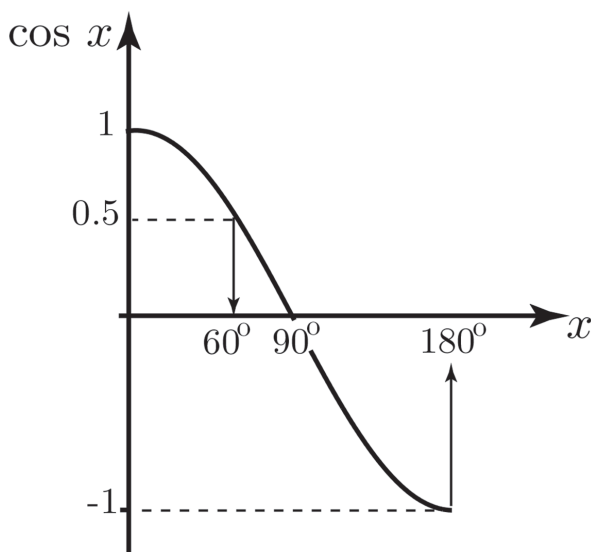


Figure 9: A graph of $\cos x$.

Example

Suppose we wish to solve the equation $3 \tan^2 x = 2 \sec^2 x + 1$ for $0^\circ \leq x \leq 180^\circ$. In this example we will simplify the equation using the identity $\sec^2 x = 1 + \tan^2 x$.

$$\begin{aligned} 3 \tan^2 x &= 2 \sec^2 x + 1 \\ 3 \tan^2 x &= 2(1 + \tan^2 x) + 1 \\ 3 \tan^2 x &= 2 + 2 \tan^2 x + 1 \end{aligned}$$

Rearranging we can write

$$\tan^2 x = 3$$

so that

$$\tan x = +\sqrt{3} \text{ or } -\sqrt{3}$$

We solve each of these equations separately.

The solutions of $\tan x = \sqrt{3}$ can be obtained by inspecting the graph in Figure 10. From the Table 1, we know that one angle with a tangent of $\sqrt{3}$ is 60° . There are no other solutions in the given interval. Using the symmetry of the graph we can deduce the solution of the equation $\tan x = -\sqrt{3}$. This is $x = 120^\circ$.

So the given equation has two solutions, $x = 60^\circ$ and $x = 120^\circ$.

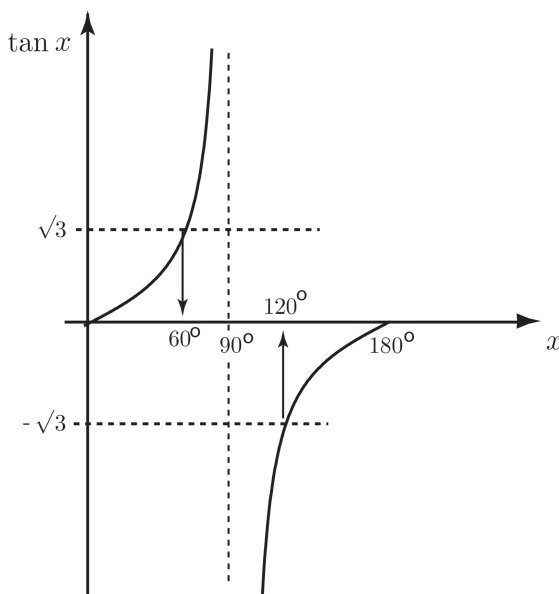


Figure 10: A graph of $\tan x$.

Exercise 2

Find all the solutions of each of the following equations in the given range

- (a) $3 \cos^2 x - 3 = \sin^2 x - 1$ for $0 < x < 360^\circ$
- (b) $5 \cos^2 x = 4 - 3 \sin^2 x$ for $-180^\circ < x < 180^\circ$
- (c) $\tan^2 x = 2 \sec^2 x - 3$ for $-180^\circ < x < 180^\circ$
- (d) $\cos^2 x + 3 \cos x = \sin^2 x - 2$ for $0 < x < 360^\circ$

Some examples where the interval is given in radians

In the previous examples, we sought solutions of equations where the angle required was measured in degrees. We now look at some examples where the angle is measured in radians. In fact, it is advisable to work with angles in radians because many trigonometric equations only make sense when an angle is measured in this way

Example

Suppose we wish to solve the equation $x = -1$ for $-\pi \leq x \leq \pi$.

A graph of the tangent function over this interval is shown in Figure 11. We know from the Table 1, that one angle with a tangent equal to 1 is $\frac{\pi}{4}$.

Using the symmetry of the graph we can deduce that the solutions of the equation $\tan x = -1$ are

$$x = -\frac{\pi}{4}, \frac{3\pi}{4}$$

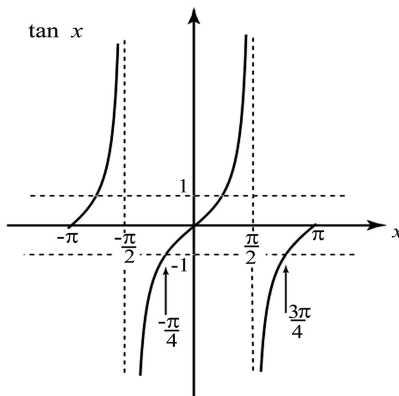


Figure 11: A graph of $\tan x$.

Example

Suppose we wish to solve the equation $\cos 2x = \frac{\sqrt{3}}{2}$ for $0 \leq x \leq 2\pi$.

We handle the multiple angle by letting $u = 2x$ so that the problem becomes that of solving

$$\cos u = \frac{\sqrt{3}}{2} \quad \text{for } 0 \leq u \leq 4\pi$$

So we have plotted a graph of $\cos u$ over this interval in Figure 12.

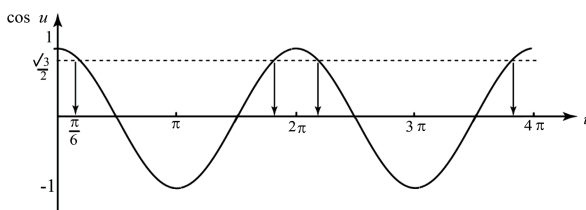


Figure 12: A graph of $\cos u$.

We know from the Table 1, that an angle which has cosine equal to $\frac{\sqrt{3}}{2}$ is 30° , that is $u = \frac{\pi}{6}$. This is indicated on the graph. Using the symmetry of the graph we can deduce all the angles which have cosine equal to $\frac{\sqrt{3}}{2}$. These are

$$u = \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6}$$

But $u = 2x$ and so

$$\begin{aligned} 2x &= \frac{\pi}{6}, 2\pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi - \frac{\pi}{6} \\ &= \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

and so

$$x = \frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$$

CONCEPT OF EXTENSION TO THE COMPLEX PLANE

Since the inverse trigonometric functions are analytic functions, they can be extended from the real line to the complex plane. This results in functions with multiple sheets and branch points. One possible way of defining the extension is:

$$\arctan(z) = \int_0^z \frac{dx}{1+x^2} \quad z \neq -i, +i$$

where the part of the imaginary axis which does not lie strictly between the branch points $(-i$ and $+i)$ is the branch cut between the principal sheet and other sheets. The path of the integral must not cross a branch cut. For z not on a branch cut, a straight line path from 0 to z is such a path. For z on a branch cut, the path must approach from $\text{Re}[x] > 0$ for the upper branch cut and from $\text{Re}[x] < 0$ for the lower branch cut.

The arcsine function may then be defined as:

$$\arcsin(z) = \arctan\left(\frac{z}{\sqrt{1-z^2}}\right) \quad z \neq -1, +1$$

where (the square-root function has its cut along the negative real axis and) the part of the real axis which does not lie strictly between -1 and $+1$ is the branch cut between the principal sheet of $\arcsin$ and other sheets;

$$\arccos(z) = \frac{\pi}{2} - \arcsin(z) \quad z \neq -1, +1$$

which has the same cut as $\arcsin$;

$$\text{arccot}(z) = \frac{\pi}{2} - \arctan(z) \quad z \neq -i, i$$

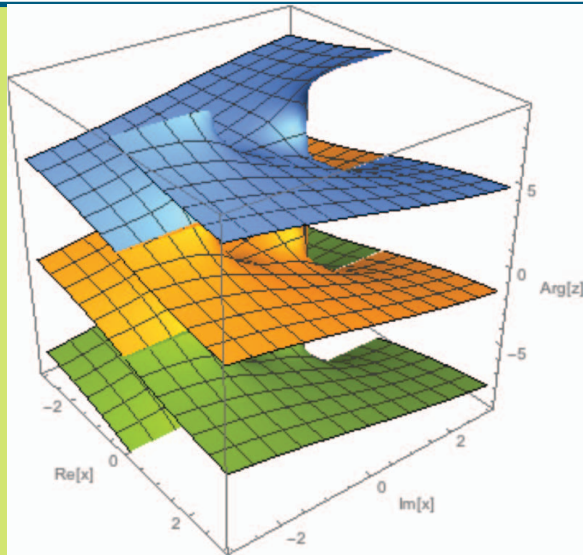
which has the same cut as $\arctan$;

$$\text{arcsec}(z) = \arccos\left(\frac{1}{z}\right) \quad z \neq -1, 0, +1$$

where the part of the real axis between -1 and $+1$ inclusive is the cut between the principal sheet of arcsec and other sheets;

$$\text{arccsc}(z) = \arcsin\left(\frac{1}{z}\right) \quad z \neq -1, 0, +1$$

which has the same cut as arcsec .



SUMMARY

- Inverse trigonometric functions are defined as the inverse functions of the basic trigonometric functions, which are sine, cosine, tangent, cotangent, secant and cosecant functions.
- The inverse trigonometric functions are essential for solving trigonometric equations and for dealing with various problems involving angles and triangles.
- Inverse trigonometric functions are also called “arc functions” since, for a given value of trigonometric functions, they produce the length of arc needed to obtain that particular value.
- The inverse trigonometric functions perform the opposite operation of the trigonometric functions such as sine, cosine, tangent, cosecant, secant and cotangent.
- In modern mathematics, there are six basic trigonometric functions: sine, cosine, tangent, secant, cosecant and cotangent. The inverse of these functions are inverse sine, inverse cosine, inverse tangent, inverse secant, inverse cosecant and inverse cotangent.
- The inverse trigonometric functions are used to determine the angle measure when at least two sides of a right triangle are known.
- There are two methods for determining an inverse trigonometric function. The first is by using a table containing all the results for every ratio. It can be tedious and cumbersome. The other is using a scientific calculator. The inverse functions for the sine, cosine and tangent can be determined quickly.

MULTIPLE CHOICE QUESTIONS

1. The domain of $\sin^{-1}(2x)$ is
 - a. $[0, 1]$
 - b. $[-1, 1]$
 - c. $[-1/2, 1/2]$
 - d. $[-2, 2]$
2. If $\sin^{-1}x + \sin^{-1}y = \pi/2$, then value of $\cos^{-1}x + \cos^{-1}y$ is
 - a. $\pi/2$
 - b. π
 - c. 0
 - d. $2\pi/3$
3. Which of the following is the principal value branch of $\cos^{-1}x$?
 - a. $[-\pi/2, \pi/2]$
 - b. $(0, \pi)$
 - c. $[0, \pi]$
 - d. $(0, \pi) - \{\pi/2\}$
4. The value of the expression $\sin [\cot^{-1} (\cos (\tan^{-1} 1))]$ is
 - a. 0
 - b. 1
 - c. $1/\sqrt{3}$
 - d. $\sqrt{(2/3)}$
5. The domain of $y = \cos^{-1} (x^2 - 4)$ is
 - a. $[3, 5]$
 - b. $[0, \pi]$
 - c. $[-\sqrt{5}, -\sqrt{3}] \cap [-\sqrt{5}, \sqrt{3}]$
 - d. $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$

Answer

1. (d) 2. (c) 3. (b) 4. (d) 5. (b)

REVIEW QUESTIONS

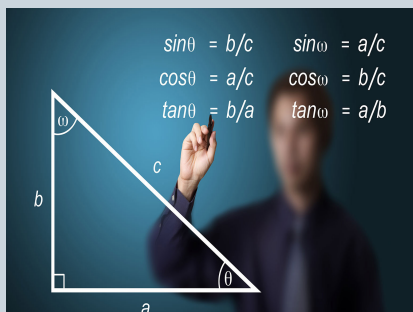
1. Solve this equation of $\tan x = \sqrt{3}$ for $0 < x < 2\pi$.
2. What are the solution of this equation: $\sin 2x = -1$ for $-\pi < x < \pi$?
3. Find all the solutions of each of the following equations in the given range.
 - (a) $3 \tan^2 x - 2 = 5 \sec^2 x - 9$ for $0 < x < 2\pi$
 - (b) $3 \cos^2 x - 6 \cos x = \sin^2 x - 3$ for $-\pi < x < \pi$
4. Find all the solutions of each of the following equations in the given range.
 - $\cos 3x = \frac{1}{\sqrt{2}}$ for $-\pi < x < \pi$
 - $\tan \frac{1}{2}x = -1$ for $-2\pi < x < 0$
5. Find all the solutions of each of the following equations in the given range.
 - $\sin x = \frac{1}{\sqrt{2}}$ for $0 < x < 360^\circ$
 - $\cos x = -\frac{1}{\sqrt{2}}$ for $0 < x < 360^\circ$

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CHAPTER 6

APPLICATIONS OF TRIGONOMETRY



Trigonometry is most simply associated with planar right-angle triangles (each of which is a two-dimensional triangle with one angle equal to 90 degrees). The applicability to non-right-angle triangles exists, but, since any non-right-angle triangle (on a flat plane) can be bisected to create two right-angle triangles, most problems can be reduced to calculations on right-angle triangles.

After studying this chapter, you will understand the core concepts of:

- Applications and Jobs that Use Trigonometry
- Law of Sines
- Law of Cosines

INTRODUCTION

Applications of trigonometry can be found in different fields in our day-to-day lives. One of the well-known fields is astronomy where trigonometry helps in determining the distances of the Earth from the planets and stars. It is used in constructing maps in geography and navigation. Applications of trigonometry are also used in finding an island's position in relation to the longitudes and latitudes. Even today, some of the technologically advanced methods which are used in engineering and physical sciences are based on the concepts of trigonometry.

Applications of trigonometry are applied in areas such as architecture, celestial mechanics, surveying, etc. The most common fields are astronomy and physics where it helps in finding the distance between the stars and planets, the path in motion, and analyzing the waves. Some of the applications include:

- Various fields like oceanography, seismology, meteorology, physical sciences, astronomy, acoustics, navigation, electronics, and many more.
- It is also helpful to find the distance of long rivers, measure the height of the mountain, etc.
- Spherical trigonometry has been used for locating solar, lunar, and stellar positions.

Trigonometry in Astronomy

Astronomy is one of the most popular fields that use trigonometry as one of the essential applications for the basic functioning of the field. Trigonometric tables were created over decades ago for computations in this field. It helps in determining the distance between the stars and planets. The tables help in locating the position of a sphere and this kind of trigonometry is called spherical trigonometry. This type of trigonometry is called linear algebra and is taught to children at a young age.

Trigonometry in Flight Engineers

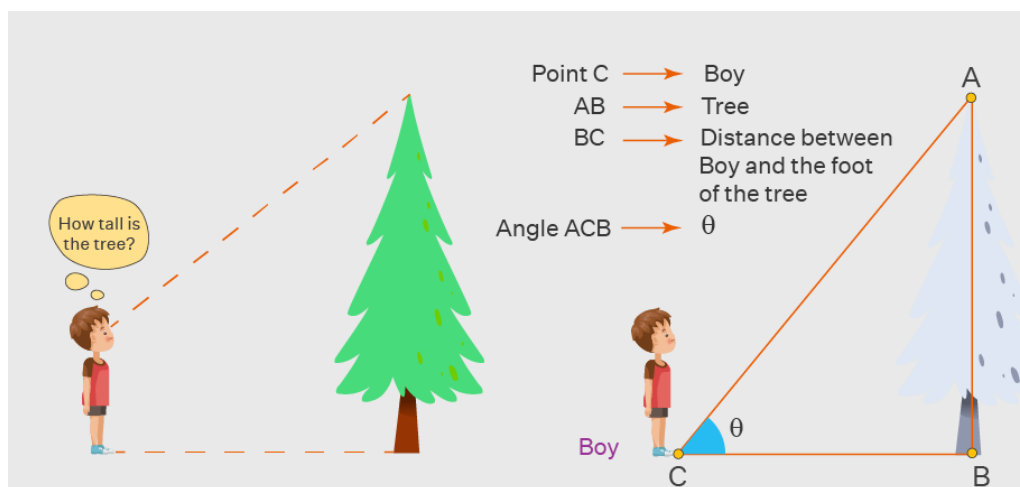
There has been significant growth in aviation or flight engineers in terms of functionality because of trigonometry. It helps in determining the speed, direction, and distance of an airplane along with keeping in mind the speed and direction of the wind. The wind plays an important role while landing and taking off to calculate at what angle and what speed is the best for that airplane.

Trigonometry in Navigation

Another use of trigonometry is seen in navigating directions with the help of a compass that provides the straight direction to a destination. Here, trigonometry functions come into play that makes it simple to find the location and also find the distance. The satellite navigation system also uses trigonometric functions such as the law of cosine to calculate simple equations to help in locating the satellites in the earth's orbit 24 hours.

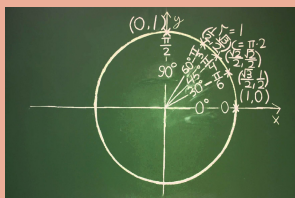
Trigonometry in Measuring Heights

Calculating the height of a mountain or a tall building can be easily found by trigonometry. The height can be determined by measuring the horizontal distance from the base and finding the angle of elevation to the top. This formation will represent a right-angle triangle with three sides, perpendicular, base, and hypotenuse. These three aspects are the necessities of calculating any trigonometric equations. Let us look at an example to understand the measurement of height. A boy is standing near a tree. He looks up at the tree and wonders “How tall is the tree?” The height of the tree can be found without actually measuring it. What we have here is a right-angled triangle, i.e., a triangle with one of the angles equal to 90 degrees. Trigonometric formulas can be applied to calculate the height of the tree, if the distance between the tree and boy, and the angle formed when the tree is viewed from the ground is given.



Trigonometry in Construction

Trigonometry is used in construction sites in measuring ground surfaces and fields. When constructing a building, aspects like roof inclination, roof slopes, perpendicular and parallel walls, light angles, sun shading, etc., require trigonometry. It also helps while installing ceramic tiles and stones while construction.



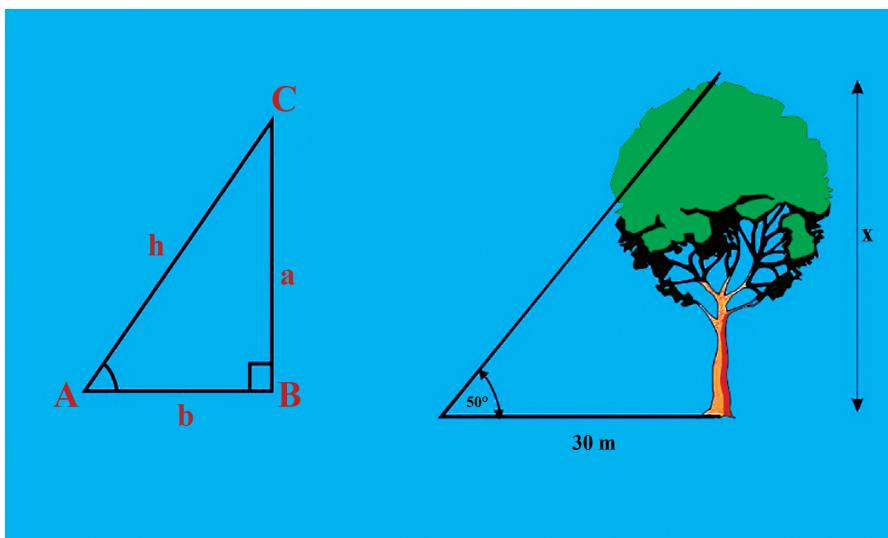
CORE CONCEPT: APPLICATIONS AND JOBS THAT USE TRIGONOMETRY

Trigonometry simply means calculations with triangles (that's where the tri comes from). It is a study of relationships in mathematics involving lengths, heights and angles of different triangles. The field emerged during the 3rd century BC, from applications of geometry to astronomical studies. Trigonometry spreads its applications into various fields such as architects, surveyors, astronauts, physicists, engineers and even crime scene investigators.

Trigonometry can be simply defined as calculations with triangles involved with the study of lengths, heights and angles. Trigonometry and its functions have an enormous number of uses in our daily life. For instance, it is used in geography to measure the distance between landmarks, in astronomy to measure the distance of nearby stars and also in satellite navigation system.

Trigonometry is a study of the relationship of angles, lengths and heights. It was emerged in the third century BC involving applications from astronomy to geometric studies. Now it has spread its application into wider fields like engineering, physics, surveyors, architects, astronauts and even in the investigation of a crime scene.

Apart from astronomy and geography, trigonometry is applicable in various fields like satellite navigation, developing computer music, chemistry number theory, medical imaging, electronics, electrical engineering, civil engineering, architecture, mechanical engineering, oceanography, seismology, phonetics, image compression and game development.



The immediate answer expected would be mathematics but it doesn't stop there even physics uses a lot of concepts of trigonometry. Another answer According to Morris Kline, in his book named- Mathematical Thought from Ancient to Modern Times, proclaimed that 'trigonometry was first developed in connection with astronomy, with applications to navigation and construction of calendars. This was around 2000 years ago. Geometry is much older, and trigonometry is built upon geometry'. However, the origins of trigonometry can be traced to the civilizations of ancient Egypt, Mesopotamia and India more than 4000 years ago.

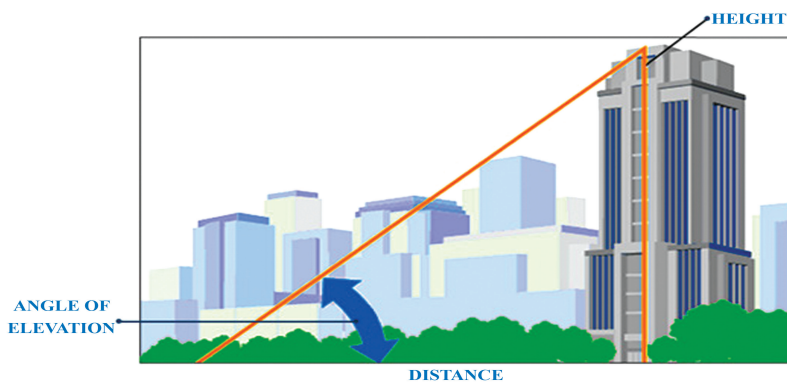
Trigonometry may not have its direct applications in solving practical issues, but it is used in various things that we enjoy so much. For example music, as you know sound travels in waves and this pattern though not as regular as a sine or cosine function, is still useful in developing computer music. A computer cannot obviously listen to and comprehend music as we do, so computers represent it mathematically by its constituent sound waves. And this means sound engineers need to know at least the basics of trigonometry. And the good music that these sound engineers produce is used to calm us from our hectic, stress full life – All thanks to trigonometry.

It may not have direct applications in solving practical issues, but used in various field. For example, trigonometry is used in developing computer music: as you are familiar that sound travels in the form of waves and this wave pattern through a sine or cosine function for developing computer music. Here are few applications where trigonometry and its functions are applicable.

Trigonometry to Measure Height of a building or a mountain:

If you know the distance from where you observe the building and the angle of elevation you can easily find the height of the building. Similarly, if you have the value of one side and the angle of depression from the top of the building you can find another side in the triangle, all you need to know is one side and angle of the triangle.

Trigonometry is used to in measuring the height of a building or a mountain. The distance of a building from the viewpoint and the elevation angle can easily determine the height of a building using the trigonometric functions.



Example: The distance from where the building is observed is 90ft from its base and the angle of elevation to the top of the building is 35° . Now find the height of the building.

Solution:

Given:

- Distance from the building is 90feet from the building.
- The angle of elevation from to the top of the building is 35° .

To solve and find the height of the tower by recalling the trigonometric formulas. Here, the angle and the adjacent side length are provided. So, using the formula of $\tan$.

$$\tan 35^\circ = \frac{\text{Opposite Side}}{\text{Adjacent Side}}$$

$$\tan 35^\circ = h/90$$

$$h = 90 \times \tan 35^\circ$$

$$h = 90 \times 0.4738$$

$$h = 42.64 \text{ feet}$$

Thus, the height of the building is 42.64feet.

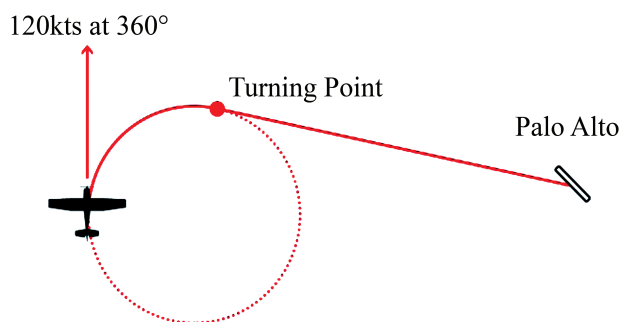
Trigonometry is used in aviation extensively, both in the calculations performed by the machines and computers used by the pilots, and by pilots performing quick rudimentary calculations and estimates themselves. Trigonometry and trigonometric functions are used to estimate distances and landing patterns and navigate around obstacles.

Trigonometry in Aviation

The aviation technology has been evolved in many up-gradations in the last few years. It has taken in account that the speed, direction, and distance as well as has to consider the speed and direction of the wind.

The wind plays a vital role in when and how a flight will travel. This equation can be solved by using trigonometry.

For example, if an airplane is traveling at 250 miles per hour, 55° of a north of east and the wind blowing due to south at 19 miles per hour. This calculation will be solved using the trigonometry and find the third side of the triangle that will lead the aircraft in the right direction.



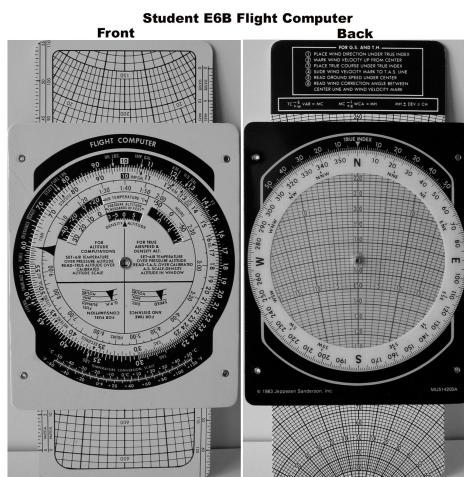
Although the extent to which trigonometry is used by pilots and other navigators has dwindled over the years, many of their methods and computer systems are

heavily ground in trigonometry and advanced mathematics. GPS, which can be found aboard nearly every modern air vehicle, uses the principles of trigonometry to compute distances and routes between locations. Trigonometry allows pilots and flight engineers to account for their speed, velocity, angle of ascent or descent, as well as other external factors such as wind speed and direction. To do this, they utilize triangles and other structures composed of vectors. Computations involving these vectors allow them to solve complex problems and accurately model the behavior of their vehicles and environments.

Many pilots are also trained in the use of on-the-spot trigonometry to come up with quick estimates of their location and positioning in the air space. Usually, these calculations are based on the angle and position of the sun relative to a pilot's plane.

Trigonometry use by Commercial Pilots

Pilots use trigonometry primarily for calculations involving wind angles, either for navigation or for cross-wind components for takeoff or landing. Note that pilots often accomplish these calculations with a flight computer such as the E6B, a circular slide rule:



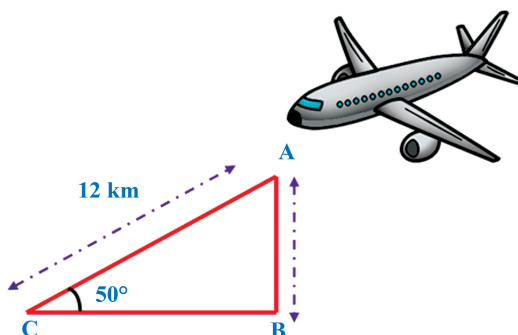
Many such flight computers are available, including electronic versions, such as the Sporty's electronic E6B. Personally, I now prefer to use web-based software, but in flight school my preferred calculation method was a TI-83 programmed to do vector addition and other pre-flight calculations. In the cockpit, the CR-3—a pocket version of the larger E6B mechanical flight computer and which can be operated with one hand—is usually the most functional method.

Regardless of how they are performed, such calculations are typically for the purpose of either:

- Calculating wind correction angles for cross-country navigation based on forecast wind conditions, or
- Calculating crosswind component for takeoff or landing based on reported wind conditions.

Navigation is a field of study that focuses on the process of monitoring and controlling the movement of a craft or vehicle from one place to another.

The latter are fairly straight-forward; the former are more complex. Crosswind component is a simple tangent calculation. Wind correction angles are more complex involving vector addition taking into account magnetic course, true airspeed, forecast wind angle, and forecast wind speed.



Now, the original question was merely, do pilots need to use trigonometry. The answer to that question is, “Yes”. The question has since been edited to specify the use of trigonometry “directly, with trigonometry knowledge”. The answer to that question is, “No”, pilots do not need to have a working understanding of trigonometry, though they do need to be capable of computing the above described wind problems. A pilot need not understand the trigonometry used by the calculator, slide rule, or FMS that he or she uses to perform these calculations. A good pilots who have no real working knowledge of trigonometry and do quite well without it.

For a pilot flying in the IFR environment, wind correction calculation is less frequently a needed calculation since most navigational means provide for wind correction in some way. This is especially true of GPS or RNAV **navigation**, because the ground track can be compared to the aircraft’s heading to show **wind correction angle**.

Trigonometry in Criminology

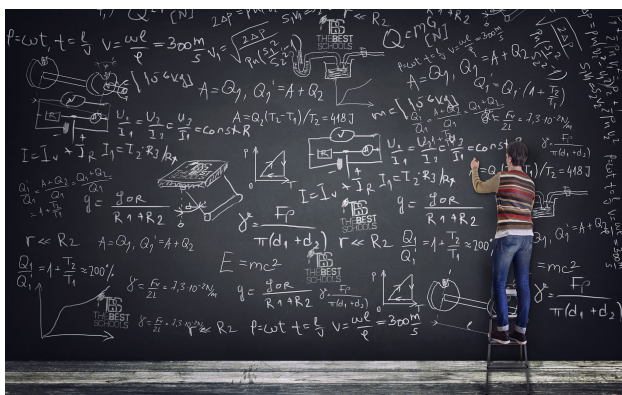
Trigonometry is even used in the investigation of a crime scene. The functions of trigonometry are helpful to calculate a trajectory of a projectile and to estimate the causes of a collision in a car accident. Further, it is used to identify how an object falls or in what angle the gun is shot.

Criminology is the study of the causes and consequences of criminal behaviors. When a bullet is shot out of a pistol and makes contact with its target, it leaves a mark at a certain angle. That angle and the size of the specific bullet and gun can be

used to trace the projectiles trajectory to estimate the location of the shooter. Other than homicide detectives and forensics agents, this knowledge is also useful for snipers who take shots over long distances.

Trigonometry is just as useful in the field of criminal justice as it would be to an engineer. Trigonometry could be used to help uncover valuable clues, leading to breaks in cases, and also ensure that those in law enforcement are meeting educational standards. As a prerequisite and in practice, trigonometry helps add to and regulate many divisions and sub-divisions of criminal justice. As a vital part of mathematics, trigonometry can end up being just as useful as a flashlight in a dimly-lit area. Let's explore these and other scenarios with three core examples of how trigonometry helps criminal justice.

Crime scene investigation dramas on television are all the rage now, so let's look at a highly-used scenario that, surprisingly, has an extremely trigonometric base. Suppose an old man fell off a ledge of a building in an apparent suicide, with a suicide note and everything. The impact marks on the ground where the man fell are abnormal for someone his age. This suggests that the speed of the man when he fell was also highly abnormal, with the rate of increase in speed not making sense for someone so old and with barely enough power to jump off the ledge. Not only that, the distance he flew from the edge of the building into the open air is also abnormal for someone such as himself.



By using the distance between the ledge of the building and where the man fell as one “leg” and the distance from the ledge to the ground as another, with respect to the speed of the man as he continued to fall, crime scene investigators could use the Pythagorean Theorem, effectively turning a “hunch” into probable cause: upon use of this trigonometric formula, it

Wind correction angle is the correction angle applied to the aircraft's course by pointing its nose towards the wind in such a way that it counteracts the effect of wind.

is mathematically proven that the man couldn't have pushed himself that far off the ledge, and wouldn't have fallen at as high a speed as he did. Instead of an apparent suicide, crime scene investigators now have a reasonable suspicion that the man was murdered: someone pushed him off the building, since it's mathematically proven that he simply couldn't have done it himself.

Trigonometry can also help turn the tide in civil suits. Let's say a man hired a contractor to help refurbish his beachfront home with extra foundational support. Beachfront homes rely basically on wooden "legs" that keep the house upright. After completing work and accepting payment, the contractor is sued under false pretenses that his work was done shabbily, as the house ended up falling down due to a lack of support, hurting the man and breaking some of his possessions. Suppose the accuser's lawyers point out that the spacing of the "legs" was too wide, and ended up being unable to support the weight of the house. By using the "legs" of the house as "legs" of a triangle and the distance between them as the hypotenuse, the contractor could prove that his work wasn't at fault. Suppose the average weight support needed to keep the house erect at a certain point was 1,000 pounds. By using the sine and cosine functions, the "legs" at that point as legs of a triangle, and the distance between them as the hypotenuse, the man discovers that both legs, combined, supported the 1,000 pounds at that one point. He could then have reasonable evidence to delay the suit and collect further evidence on all of the additional "legs". In time and with the proper evidence, mathematically the contractor can prove without the slightest hint of a doubt that his work was flawless.

Now, let's consider something that has become all the rage in cell phones and vehicles: global positioning devices. The mighty "GPS" is little more than a satellite navigational system. And, despite most not knowing it, a G.P.S. system and others like it are entirely based on trigonometric formulas and calculations, the principle of which is a process known as "Triangulation". Using Triangulation, law enforcement officers could pursue a criminal across the country, and it is even possible that G.P.S. systems could be surgically implanted into hardened convicts, so police and parole officers will always know where they are. In a nutshell, effective and exact Triangulation is the difference between arriving at an abandoned New York home just in time to stop an escaped criminal from committing murder as compared to showing up with guns-blazing at a New Jersey daycare.

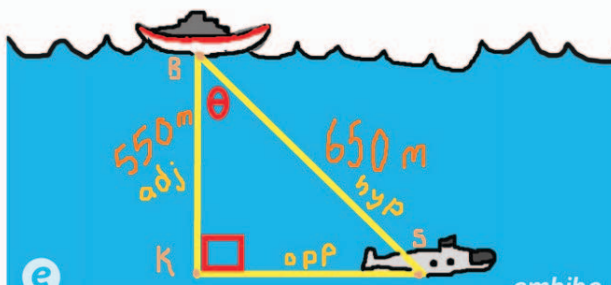
There are many other ways that trigonometry affects criminal justice, most all of them beneficial. Most positions of law enforcement require a certain level of mathematical skill, for example. Usually, this means that unless you've taken a college course on trigonometry and other related mathematics; don't expect to immediately be put on the payroll of an established law enforcement team, unit, or division. Also and quite literally, trigonometry could be the thing that saves your life: if you calculate the effective lethal range of bullets during a shoot-out, you could take prisoners by surprise and astonish them by not backing down...or, if you miscalculate, you could wind up getting yourself shot and killed. Like any field of mathematics, trigonometry is only useful if you are exact. But if you are, you'll be able to figure out all sorts

of things that have as many uses as you could think of, especially in the field of criminal justice. From proving yourself innocent to triangulating the exact location of a felon, there's very little that trigonometry can't do in the field of criminal justice. In criminology, trigonometry can help to calculate a projectile's trajectory, to estimate what might have caused a collision in a car accident or how did an object fall down from somewhere, or in which angle was a bullet shot etc.

Trigonometry in Marine Biology

Trigonometry is often used by marine biologists for measurements to figure out the depth of sunlight that affects algae to photosynthesis. Using the trigonometric function and mathematical models, marine biologists estimate the size of larger animals like whales and also understand their behaviors.

In marine engineering trigonometry is used to build and navigate marine vessels. To be more specific trigonometry is used to design the Marine ramp, which is a sloping surface to connect lower and higher level areas, it can be a slope or even a staircase depending on its application



Marine biologists often use trigonometry to establish measurements. For example, to find out how light levels at different depths affect the ability of algae to photosynthesize. Trigonometry is used in finding the distance between celestial bodies. Also, marine biologists utilize mathematical models to measure and understand sea animals and their behavior. Marine biologists may use trigonometry to determine the size of wild animals from a distance.

Marine biologists use trigonometry and algebra to establish measurements. They also use math for measurement equivalents, as they frequently need to convert between standard and metric units of measurement.

Because math is the essential language of science, marine biologists use math to solve a number of complex problems in their line of work. These problems require the need for the biologists to collect and analyze data in a variety of forms. Marine biologists use the data to make theoretical models and predictions, and that information is sometimes used to establish policy changes. Sometimes marine biologists have tasks that require solutions to be prepared at a number of different concentrations, and solving those problems requires the skill of balancing equations and algebra.

Anyone studying marine biology in college is required to take a number of different math courses, such as algebra and trigonometry, which are helpful in establishing

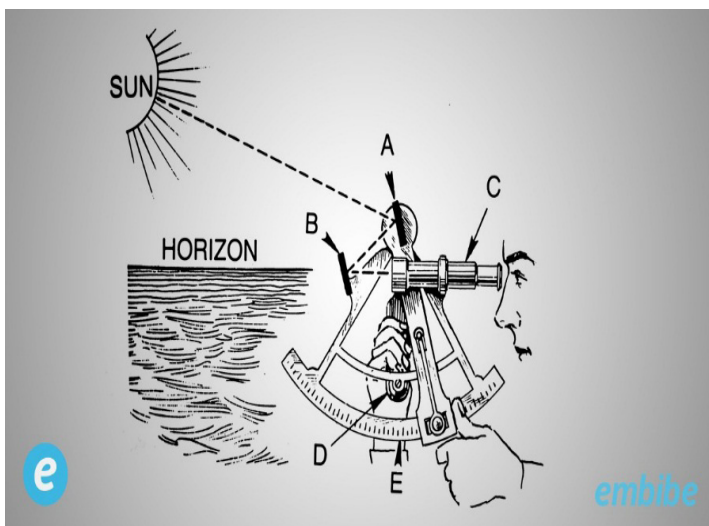
measurement. Some colleges require the successful completion of algebra, trigonometry and possibly elementary analysis or pre-calculus before enrollment in a calculus course. Calculus helps with data analysis and mathematical modeling. Some colleges require those studying marine biology to take at least one statistics course. Marine biologists utilize statistics to test hypotheses and mathematical models.

Trigonometry in Navigation

Trigonometry is used in navigating directions; it estimates in what direction to place the compass to get a straight direction. With the help of a compass and trigonometric functions in navigation, it will help to pinpoint a location and also to find distance as well to see the horizon.

Trigonometry is used to set directions such as the north south east west, it tells you what direction to take with the compass to get on a straight direction. It is used in navigation in order to pinpoint a location. It is also used to find the distance of the shore from a point in the sea. It is also used to see the horizon.

Cartesian coordinate is a coordinate system that specifies each point uniquely in a plane by a pair of numerical coordinates, which are the signed distances to the point from two fixed perpendicular directed lines, measured in the same unit of length.



Navigation is the process to accurately determine one's position and planning a route for the movement of a craft or vehicle from one place to another. Navigation also refers to the specialized knowledge used by navigators to perform navigational tasks. Navigational techniques involve locating positions compared to known locations and patterns as well as determining the distance between places. Modern navigation rely on satellites to determine positions and to travel by sea and by air.

Trigonometry is fundamental to navigation. Trigonometry is used extensively in calculations referring to **Cartesian coordinates**. Cartesian coordinates are used to represent North, South, and East and West directions. Through the use of Trigonometry the distance between objects, the required direction and a vessel's or aircraft's bearings can be calculated.

Compasses and Direction

A compass is a navigational instrument that shows four direction points. These points are North, South, East and West. A compass rose, showing these directions is drawn on the compass with 'N' signifying North. Compasses are essential in Navigation for it determines bearings and which way to travel. Before the compass, the position and direction at sea was done by the sighting of landmarks and celestial bodies (stars).

The invention of the compass enabled a heading to be determined when bad weather occurred in addition to latitude and longitude. Without the compass people could not accurately navigate away from land and therefore not engage in trade with other nations.

What are bearings?

In marine navigation, a bearing is the direction one object is from another object, usually, the direction of an object from one's own vessel. In aircraft navigation, a bearing is the actual compass direction of the forward course of the aircraft. In land navigation, a bearing is the angle between a line connecting two points. Bearings is usually measured in degrees.

Types of bearings include:

- True bearings/ Three-figure bearings
- Magnetic bearings
- Compass bearings
- Grid bearings

Trigonometry in action

A plane leaves an airport and flies 600 km on a bearing of 145°

a) How far south of the airport is the plane?

Let x = distance south and missing angle in SAP = 35°

$$\cos 35^\circ = \frac{x}{600}$$

$$600$$

$$x = 600 \cos 35^\circ$$

$$x = 491 \text{ Km}$$

Therefore the plane is 491 km south of the airport

b) What is the three-figure bearing of the airport from the plane?

A compass rose is drawn at the position of the plane.

NPA = 35° (alternate)

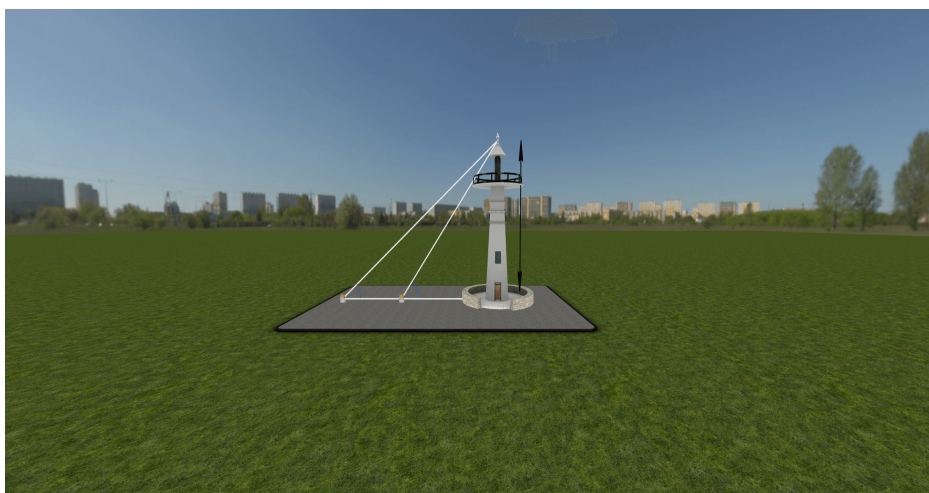
Therefore the bearing of the airport from the plane is $360^\circ - 35^\circ = 325^\circ$

Math Review of Trigonometry and Navigation

Trigonometry in the form of triangulation is at the heart of navigation, whether it is by land, sea, or air. GPS systems use triangulation to find and fix positions, extrapolating what is unknown from what is known. Triangulation is also used to measure the distances between earth and distant stars and galaxies, thus aiding future space travel.

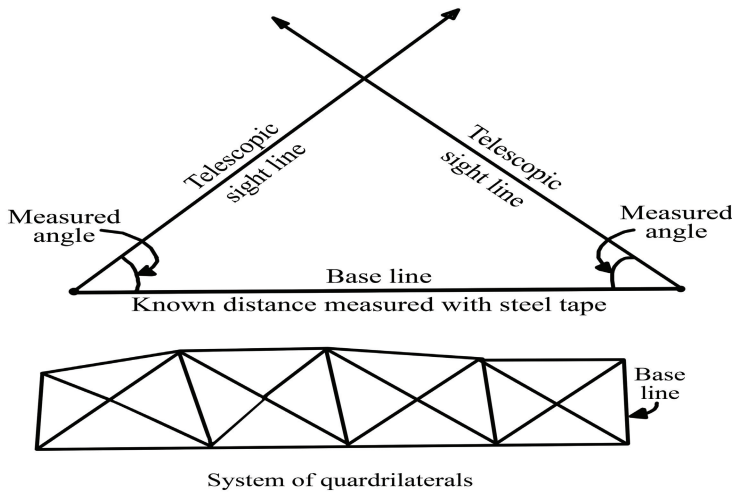
Trigonometry on Land

The ancient Greeks, Egyptians, and other ancient civilizations developed methods to measure triangles accurately. The Egyptians used principles of trigonometry to build the Pyramids, while the Greeks developed extensive geometric and trigonometric proofs and applied them to many surveying and navigational problems. If the measurement of two angles are known, the third can always be calculated. The tangent of a right triangle can also be used to calculate valuable distances such as the height of a tall tree or of a mountain.



Trigonometry and Surveying

Surveyors use a special type of telescope called a theodolite. It is mounted on a tripod, and measures distances and angles in three-dimensional space. Mounting a telescope on a tripod allows the telescope to be rotated, making very precise measurements possible. Known distances and angles can then be triangulated for accurate mapmaking, road building, and other engineering applications. The procedure for triangulation is similar to measuring the height of a tree. Suppose one knows the accurate measure of baseline AB. The distance to point C can be used by measuring the angles for AC and BC very accurately and using the trigonometric ratios.



Navigation by Sea

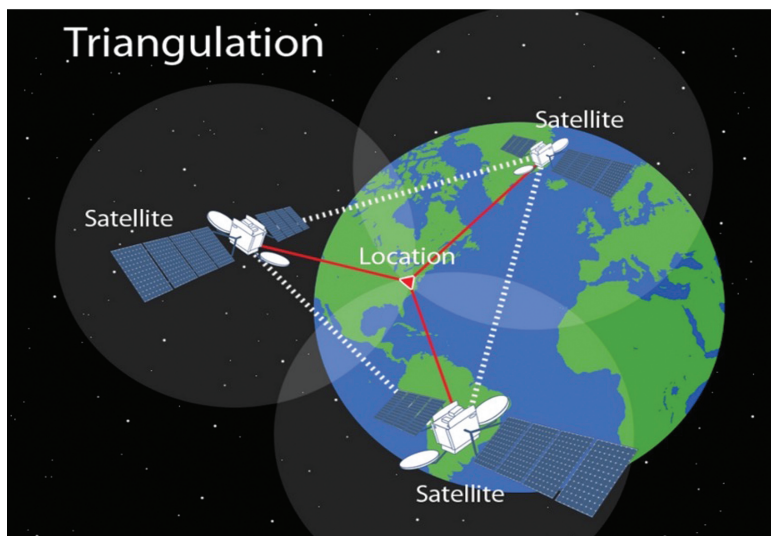
Navigation by sea is complicated by large distances without landmarks in open ocean. The principles of trigonometry and triangulation apply. For most of the time humanity has moved through water for long distances, the only landmarks have been the positions of the sun by day and the stars by night. Those angles and distances can be measured accurately by using devices such as the marine sextant (for angles) and the chronometer (for the exact time that measurements are taken). Navigation by sea is based upon spherical **trigonometry**. The exact position of a ship can be determined by the angle the celestial body makes with the horizon, measured at a precise time. The angle and precise time measurements are compared with tables of known values.

Triangulation is the process of determining the location of a point by forming triangles to the point from known points.

GPS Navigation

GPS is short for **Global Positioning System**. It has grown from an original network of 24 satellites to a network of over 30 satellites from the United States. Similar systems are under development from Russia, China, Japan, the European Union, and India, and many satellites are fully operational. Satellites orbit over the same locations every day, and emit signals continuously giving the exact time and their location. Triangulation with particular satellites allows for precise location mapping.

Global Positioning System is a global navigation satellite system that provides geo-location and time information to a GPS receiver anywhere on or near the Earth where there is an unobstructed line of sight to four or more GPS satellites.



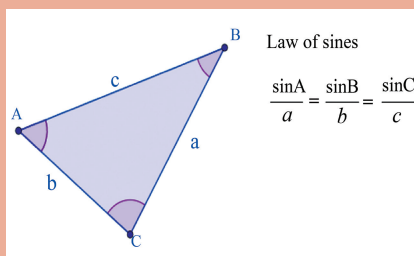
Navigation without trigonometry

Without Trigonometry there would be no bearings meaning that Navigation by sea, air and land will be different. However, the fact that Trigonometry is used in this field does not mean that the knowledge of Trigonometry is needed to learn about navigation. Some aspects of Navigation are easy to understand without Trigonometry such as directions, while other fields such as the music theory needed Trigonometry in order to fully understand how it works.

Other Uses of Trigonometry

Trigonometry is a method for relating linear measures to angular measures. Such relations are required, for example, when there are multiple force vectors acting on a system: one often needs to resolve a vector into components along a given basis, and these components depend on the angle between the given vector and the basis vectors.

- The calculus is based on trigonometry and algebra.
- The fundamental trigonometric functions like sine and cosine are used to describe the sound and light waves.
- Trigonometry is used in oceanography to calculate heights of waves and tides in oceans.
- It used in creation of maps
- It is used in satellite systems.



Law of sines

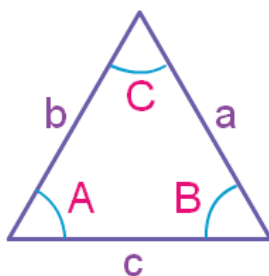
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

CORE CONCEPT: LAW OF SINES

The law of sines can be used to compute the remaining sides of a triangle when two angles and a side are known a technique known as triangulation.

Numerical calculation using this technique may result in a numerical error if an angle is close to 90 degrees. It can also be used when two sides and one of the non-enclosed angles are known. In some such cases, the triangle is not uniquely determined by this data and the technique gives two possible values for the enclosed angle.

Law of Sines Formula is used to calculate the unknown sides and angles of a triangle. The sine law states that the ratio of any side to the sine of the angle opposite to that side, is the same for all three sides. Given two sides and two angles of the triangle, the unknown side and angle can be calculated using the sine law.



The Law of Sines Formula is,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = d$$

where a , b , and c are the lengths of the sides of a triangle, and A , B , and C are the opposite angles, while d is the diameter of the triangle's circumcircle. When the last of these equations is not used, the law is sometimes stated using the reciprocals;

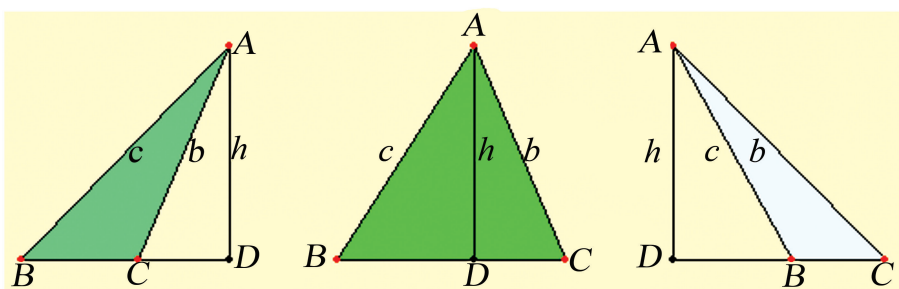
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

It's enough to show the last equation

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

since the first version differs only in the labelling of the triangle.

An explanation of the law of sines is fairly easy to follow, but in some cases we'll have to consider sines of obtuse angles.



First, drop a perpendicular line AD from A down to the base BC of the triangle. The foot D of this perpendicular will lie on the edge BC of the triangle when both angles B and C are acute. But if angle B is obtuse, then the foot D will lie on BC extended in the direction of B . Yet if angle C is obtuse, then D will line on BC extended in the direction of C . Fortunately, the argument is the same in all three cases.

Let h denote the length of this line AD , that is, the height (or altitude) of the triangle.

When angle B is acute, then $\sin B = h/c$. But this is true even when B is an obtuse angle as in the third diagram. There, angle ABC is obtuse. But the sine of an obtuse angle is the same as the sine of its supplement. That means $\sin ABC$ is the same as $\sin ABD$, that is, they both equal h/c .

Likewise, it doesn't matter whether angle C is acute or obtuse, $\sin C = h/b$ in any case.

These two equations tell us that h equals both $c \sin B$ and $b \sin C$. But from the equation $c \sin B = b \sin C$, we can easily get the law of sines:

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

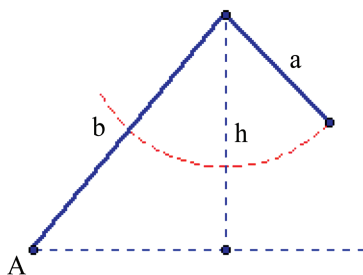
Ambiguous case of the law of Sines

Teaching this particular topic in the past has created numerous headaches for both me and my students.

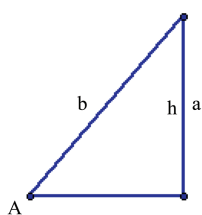
For those of you who need a reminder, the ambiguous case occurs when one uses the law of sines to determine missing measures of a triangle when given two sides and an angle opposite one of those angles (SSA). In this ambiguous case, three possible situations can occur: 1) no triangle with the given information exists, 2) one such triangle exists, or 3) two distinct triangles may be formed that satisfy the given conditions. These possibilities are summarized in the diagrams below:

Suppose we are given side a , side b and angle A of triangle ABC . Let h equal the height of the "triangle". Note that h can be found using $h = b(\sin a)$.

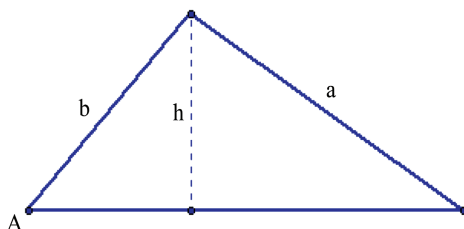
If angle A is acute, and $a < h$, no such triangle exists.



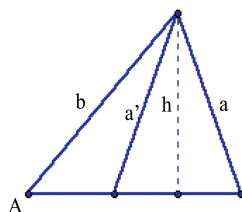
If angle A is acute, and $a = h$, one possible triangle exists.



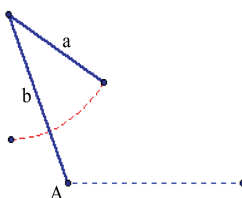
If angle A is acute, and $a > b$, one possible triangle exists.



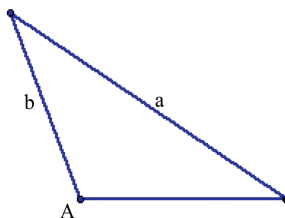
If angle A is acute, and $h < a < b$, two possible triangles exist.



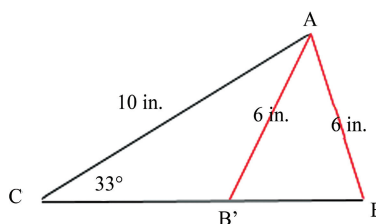
If angle A is obtuse, and $a < b$ or $a = b$, no such triangle exists.



If angle A is obtuse, and $a > b$, one such triangle exists.



When using the Law of Sines to find an unknown angle, you must watch out for the ambiguous case. This occurs when two different triangles could be created using the given information. For example, take a look at this picture:



If you are told that $m\angle C = 33^\circ$, $b = 10$ in. and $c = 6$ in, there are two different triangles that match this criteria. As you can see in the picture, either an acute triangle or an **obtuse triangle** could be created because side c could swing either in or out along the unknown side a . When you are given two sides and an angle not in between those sides, you need to be on the lookout for the ambiguous case. To determine if there is a 2nd valid angle:

Obtuse triangle is one with one obtuse angle (greater than 90°) and two acute angles. Since a triangle's angles must sum to 180° , no triangle can have more than one obtuse angle.

1. See if you are given two sides and the angle not in between (SSA). This is the situation that may have 2 possible answers.
2. Find the value of the unknown angle.
3. Once you find the value of your angle, subtract it from 180° to find the possible second angle.
4. Add the new angle to the original angle. If their sum is less than 180° , you have two valid answers. If the sum is over 180° , then the second angle is not valid.

Examples:

1. Use the Law of Sines to find the measure of angle B from our example in which $m\angle C = 33^\circ$, $b = 10$ in. and $c = 6$ in

First we know that this triangle is a candidate for the ambiguous case since we are given two sides and an angle not in

between them. We need to find the measure of angle B using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a}{\sin A} = \frac{10}{\sin B} = \frac{c}{\sin 33^\circ}$$

$$\frac{10}{\sin B} = \frac{c}{\sin 33^\circ}$$

Now we'll cross multiply and solve the equation:

$$10 (\sin 33^\circ) = 6 \cdot \sin B$$

$$\sin B = \frac{10(\sin 33^\circ)}{6}$$

$$m\angle B = \arcsin \frac{10(\sin 33^\circ)}{6}$$

$$m\angle B \approx 65.2^\circ$$

We've found one value for angle B, but we need to see if there's another possible value. To do this, we'll subtract 65.2° from 180° .

$$180^\circ - 65.2^\circ = 114.8^\circ$$

To see if 114.8° is a valid answer, we must add this to the other existing angle. If their sum is less than 180° , we know a triangle can exist. If, however, it is over 180° , then it is not a valid answer (because the three angles of a triangle must add up to 180° .)

$$114.8^\circ + 33^\circ = 147.8^\circ$$

Since 147.8° is less than 180° , we know that 114.8° is a valid answer

Final answer: $m\angle B \approx 65.2^\circ$ and $m\angle B \approx 114.8^\circ$

- Use the Law of Sines to find measure of angle A in this scenario:
 $m\angle C = 64^\circ$ $c = 10$ ft. $a = 8$ ft. If it helps, you can draw a rough sketch to view this triangle, but this is optional. We know that this triangle is a candidate for the ambiguous case since we are given two sides and an angle not in between them.

We need to find the measure of angle B using the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{8}{\sin A} = \frac{b}{\sin B} = \frac{10}{\sin 64^\circ}$$

$$\frac{8}{\sin A} = \frac{10}{\sin 64^\circ}$$

Now we'll cross multiply and solve the equation:

$$8 (\sin 64^\circ) = 10 \cdot \sin A$$

$$\sin A = \frac{8(\sin 64^\circ)}{10}$$

$$m\angle A = \arcsin \frac{8(\sin 64^\circ)}{10}$$

$$m\angle A \approx 46.0^\circ$$

We've found one value for angle A, but we need to see if there's another possible value. To do this, we'll subtract 46.0° from 180° .

$$180^\circ - 46^\circ = 134^\circ$$

To see if 134° is a valid answer, we must add this to the other existing angle. If their sum is less than 180° , we know a triangle can exist. If, however, it is over 180° , then it is not a valid answer.

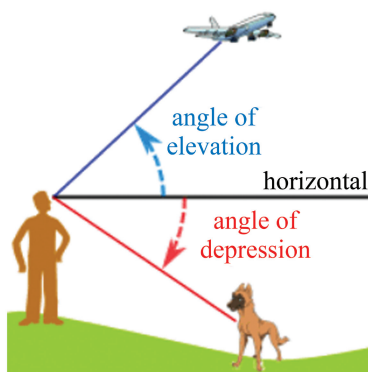
$$134^\circ + 64^\circ = 198^\circ$$

Since 198° is greater than 180° , we know the second measurement of 134° is not a valid answer.

Final answer: $m\angle A \approx 46.0^\circ$

Angle of Depression

Angle of depression is a term used to describe the angle formed by two lines, one extending from an observer's eye to the horizon and another to an object located some horizontal distance away from and below the observer. The angle of depression is a popular teaching tool in mathematics. A right triangle is formed by connecting three points, with the observer and the object serving as two of the points. The third point is located where the horizontal line from the observer to the horizon intersects with a vertical line extending upwards from the object.



If one or more of the values of the triangle, such as the length of one of the sides or the size of one of the two acute angles is not known, the angle of depression can be calculated using principles of geometry and trigonometry. These exercises are a good way to use practical, everyday situations to illustrate problems that can be difficult for some students to grasp. By creating a framework for the known and unknown values

of a problem, students are may be able to visualize the problem more effectively, which helps them to find the correct solution.

Problems involving the angle of depression assume that the line from the observer to the horizon and the ground are parallel. This is useful for situations where the distances are relatively small. When the distances are very large or are part of real world situations, however, rather than hypothetical problems, the curvature of the Earth has an effect, and certain assumptions are no longer valid, especially the one that states that the angle of elevation from the object back to the observer and the angle of depression are equal. The angle of elevation is the angle formed by the ground and a line extending from the object upwards to the observer. As long as the ground and the line extending from the observer to the horizon are parallel, the angles of depression and elevation between the observer and the object are always equal.

The angle of depression is used in surveying, engineering, and geology. Road construction, building projects and civil engineering projects may make use of the angle of depression and the concepts surrounding it to ensure precise construction of many structures as well as proper alignment of things like aqueducts and pipelines. Geologists sometimes use it to describe the arrangement of rock layers relative to the surface of the Earth.

Triangulation and the Law of Sines

A triangle has three sides and three angles; their magnitudes provide six pieces of information about the triangle. In ordinary (Euclidean) geometry, most of the time three pieces of information are sufficient to give us the other three pieces of information. In order to more easily discuss the angles and sides of a triangle, we will label the angles by capital letters (such as A, B, C) and label the sides by small letters (such as a, b, c.) Furthermore, we will assume that a side labeled with a small letter is the side opposite the angle with the same, but capitalized, letter. For example, the side a is opposite the angle A. We will also use these labels for the lengths (magnitudes) of the angles and sides, so that $a = 3$ is a reasonable abbreviation for the statement “side a has length 3.”

Three Bits of a Triangle

Suppose we have three bits of information about a triangle. Can we recover all the information? If the three bits of information are the values of the three angles, and if we have no information about lengths of sides, then the answer is No. Given a triangle with angles A, B, C (in Euclidean geometry) we can always expand (or contract) the triangle to a similar triangle which has the same angles but whose sides have all been stretched (or shrunk) by some constant factor. This “Angle-Angle-Angle” information (abbreviated AAA) is not enough information to describe the entire triangle. However, we will discover that in almost every other situation, we can recover the entire triangle.

If we know the values of two angles and one side, then, if the known side is between the two angles, we will say that we are in the Angle-Side-Angle case. (We abbreviate this case ASA.) If we know two angles and one side, but the known side

is not between the two angles, then we are in the Angle-Angle-Side (AAS) situation and we can also recover the remaining bits of the triangle. Indeed, any time we know two angles then, since the sum of the angles of a triangle is 180° , we really know all angles.

The area of an oblique triangle

An “oblique” triangle is one which is not a right triangle. Before we move on and the proof of the Law of Sines, we pause to make an observation about the area of a triangle. Since a triangle is half of a parallelogram, its area is one-half of the product of its base and height. We let K represent the area of a triangle (since we are already using the letter A for an angle.) We see that we could write c for the base of the triangle and so the area of the triangle is

$$K = \frac{1}{2}ch$$

But earlier, in our proof of the Law of Sines, we solved for h and we wrote $h = b \sin A$ and $h = a \sin B$. So we can substitute for h and write the area as

$$K = \frac{1}{2}cb \sin A \text{ or } K = \frac{1}{2}ac \sin B$$

Or we could call the known angle C and just write

$$K = \frac{1}{2}ab \sin C$$

We can summarize this by saying that the area of a triangle is one-half of the product of the sine of an angle and its neighboring sides.

Two Angles and the Law of Sines

If we know two angles of a triangle, then since the three angles add to $180^\circ = \pi$ then we can figure out the third angle. As long as we know one more bit of information, the length of a side, then the law of sines gives us the length of all sides. In our abbreviated notation for the known information, this includes the cases ASA and AAS, that is, the cases in which we know two angles and an included side or two angles and a side not between them. These cases are equivalent since we know all three angles once we know any two.

Some Worked Problems

Use the Law of Sines to solve the following triangles.

1. $A = 62^\circ$, $B = 74^\circ$, $c = 14$ feet Solution. If $A = 62^\circ$, $B = 74^\circ$, $c = 14$ feet then $C = 54^\circ$. Then, by the law of sines,

$$\frac{\sin 54^\circ}{14} = \frac{\sin 62^\circ}{a} = \frac{\sin 74^\circ}{b}$$

$$a = \frac{14 \sin 62^\circ}{\sin 54^\circ} \approx 15.28 \text{ feet}$$

2. Solve the triangle $A = 60^\circ$, $B = 70^\circ$, $c = 10$ feet

Solution. If $A = 60^\circ$, $B = 70^\circ$, $c = 10$ feet then $C = 50^\circ$ since the sum of the angles of a triangle is 180° . By the law of sines,

$$\frac{\sin 60^\circ}{a} = \frac{\sin 70^\circ}{b} = \frac{\sin 50^\circ}{10}$$

$$\text{So } a = \frac{10 \sin 60^\circ}{\sin 50^\circ} = 11.31 \text{ feet}$$

and

$$b = \frac{10 \sin 70^\circ}{\sin 50^\circ} = 12.27 \text{ feet}$$

Triangulation

One can often measure distance to an object by working out a baseline and then measuring the angles formed by lines from the distant object to the ends of the baseline. This is a classic case of ASA and is perfect for the law of sines.

A Worked Problem

Suppose that a ship is out in the harbor. Standing at the dock, a compass reveals that the ship is at a bearing of 30° east of north. But if one walks 200 yards west of the dock, the ship is now at a bearing of 40° east of north. How far is the ship from the dock?

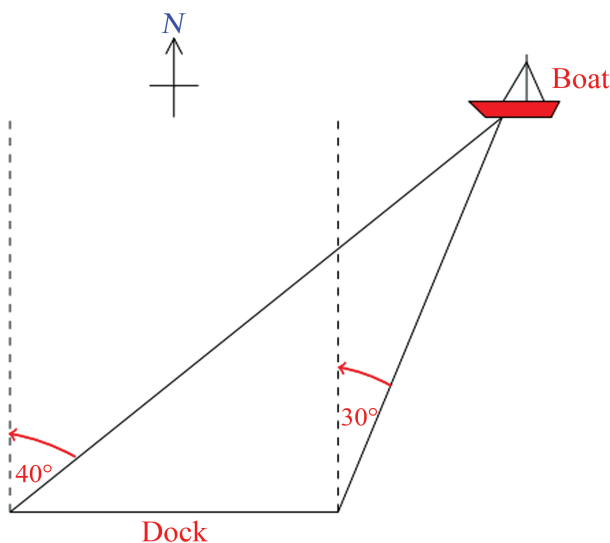
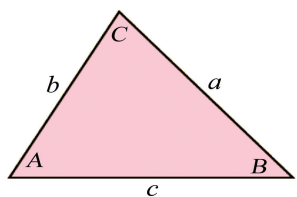


Figure 1. Measuring distance to a boat in the harbor.

If we draw an east-west baseline representing the 200 yards walked, and then draw lines from the ends of the baseline to the ship, we have a triangle in which the vertex on the west side of the baseline has an (interior) angle of $90^\circ - 40^\circ = 50^\circ$ and the vertex on the east side of the baseline has an angle of $90^\circ + 30^\circ = 120^\circ$ and so, if we call the length of the baseline $c = 200$, we have angles $A = 50^\circ$ and $B = 120^\circ$. The vertex faraway at the ship has angle 10° in this triangle (this is the difference in 40° and 30° .) Let us use C to represent the angle 10° at the vertex of this triangle where the ship sits. Then the side c is the 200 yard baseline and the angles A and B are 50° and $B = 120^\circ$. We can solve this problem by the Law of Sines and see that $a = 882$, $b = 997.45$. So the ship is 882 yards from the east end of the dock.

$$c^2 = a^2 + b^2 - 2ab \cos C$$



CORE CONCEPT: LAW OF COSINES

The law of cosines is useful for computing the third side of a triangle when two sides and their enclosed angle are known, and in computing the angles of a triangle if all three sides are known.

The law of cosines for calculating one side of a triangle when the angle opposite and the other two sides are known. Can be used in conjunction with the law of sines to find all sides and angles.

Law of cosines, Generalization of the Pythagorean Theorem relating the lengths of the sides of any triangle. If a , b , and c are the lengths of the sides and C is the angle opposite side c , then $c^2 = a^2 + b^2 - 2ab \cos C$.

There are two other versions of the law of cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

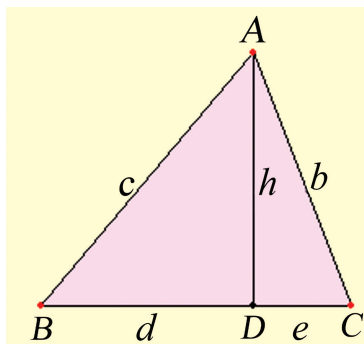
and

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Thus, the law of cosines is valid when C is an obtuse angle.

Case 1. Now consider the case when the angle at C is right. The cosine of a right angle is 0, so the law of cosines, $c^2 = a^2 + b^2 - 2ab \cos C$, simplifies to becomes the Pythagorean identity, $c^2 = a^2 + b^2$, for right triangles which we know is valid.

Case 2. In this case we assume that the angle C is an acute triangle. Drop a perpendicular line AD from A down to the base BC of the triangle. The foot D of the perpendicular will (1) lie on the edge BC if angle B is acute, (2) coincide with the point B if the angle B is right, or (3) lie on the side BC extended if the angle B is obtuse.



Let h denote the height of the triangle, let d denote BD , and e denote CD .

Then we can read the following relationships from the diagram:

$$c^2 = d^2 + h^2$$

$$b^2 = e^2 + h^2$$

$$\cos C = e/b$$

$$d^2 = (e - a)^2$$

That last equation requires explanation. If the point D lies on the side BC , then $d = a - e$, but if D lies on BC extended, then $d = e - a$. In either case, $d^2 = (e - a)^2$.

These equations and a little algebra finish the proof as follows:

$$c^2 = d^2 + h^2$$

$$= d^2 - e^2 + b^2$$

$$= (d - e)(d + e) + b^2$$

$$= (a - 2e)a + b^2$$

$$= a^2 + b^2 - 2ae$$

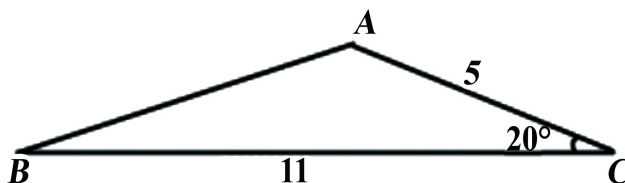
$$= a^2 + b^2 - 2ab \cos C$$

Thus, we now know that the law of cosines is valid when both angle C is acute, and we've finished all three cases.

Incidentally, Euclid included in his *Elements* a couple of propositions, II.12 and II.13, that look very much like the law of cosines, but they are not actually the law of cosines, of course, since trigonometry had not been developed in Euclid's time.

Example: Two Sides and the Included Angle-SAS

Given $a=11, b=5, m\angle C=20^\circ$. Find the remaining side and angles.



$$\begin{aligned}
 c^2 &= a^2 + b^2 - 2ab \cos C \\
 c &= \sqrt{a^2 + b^2 - 2ab \cos C} \\
 &= \sqrt{11^2 + 5^2 - 2(11)(5)(\cos 20^\circ)}
 \end{aligned}$$

$$\approx 6.53$$

To find the remaining angles, it is easiest to now use the Law of Sines.

$$\sin A \approx \frac{11 \sin 20^\circ}{6.53}$$

$$A \approx 144.82^\circ$$

$$\sin B \approx \frac{5 \sin 20^\circ}{6.53}$$

$$B \approx 15.2^\circ$$

Note that angle AA is opposite to the longest side and the triangle is not a **right triangle**. So, when you take the inverse you need to consider the obtuse angle whose sine is

$$\frac{11 \sin(20^\circ)}{6.53} \approx 0.5761$$

Right triangle is a triangle in which one angle is a right angle (that is, a 90-degree angle). The relation between the sides and angles of a right triangle is the basis for trigonometry.

Example: Three Sides-SSS

Given $a=8$, $b=19$ and $c=14$. Find the measures of the angles.



It is best to find the angle opposite the longest side first. In this case, that is side b.

$$\cos B = \frac{b^2 - a^2 - c^2}{-2ac} = \frac{19^2 - 8^2 - 14^2}{-2(8)(14)} \approx -0.45089$$

Since $\cos B$ is negative, we know that B is an obtuse angle.

$$B \approx 116.80^\circ$$

Since B is an obtuse angle and a triangle has at most one obtuse angle, we know that angle A and angle C are both acute.

To find the other two angles, it is simplest to use the Law of Sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{8}{\sin A} \approx \frac{19}{\sin 116.80^\circ} \approx \frac{14}{\sin C}$$

$$\sin A \approx \frac{8 \sin 116.80^\circ}{19}$$

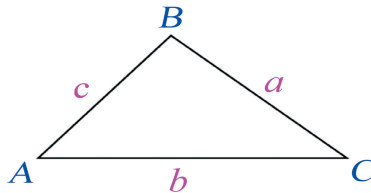
$$A \approx 22.08^\circ$$

$$\sin C \approx \frac{14 \sin 116.80^\circ}{19}$$

$$C \approx 41.12^\circ$$

Part A: The Setup and the Law

Conventional setup for a triangle:



Observe that Side a “faces” Angle A, b faces B, and c faces C.

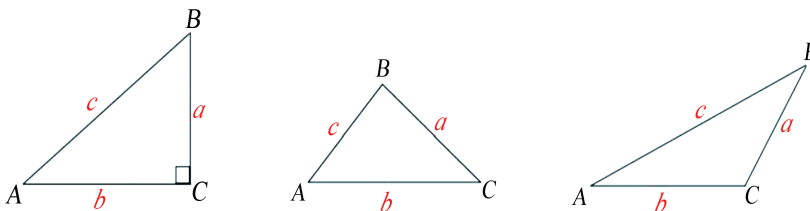
For such a triangle:

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Think / For Memorization Purposes: This looks like the Pythagorean Theorem, except that there is a third term. “The square of one side equals the sum of the squares of the other two sides, minus twice their product times the cosine of the angle included between them.” Notice that the formula is symmetric in a and b; we have 2ab in the formula as opposed to 2bc or 2ac. Angle C is the one we take the cosine of, because it is the “special” angle that faces the side indicated on the left.

Part B: A Thought Experiment

How does the formula gibe with our geometric intuition?



Let's say we fix lengths a and b, but we allow the other parts to vary. Imagine rotating the side labeled a about the point C so that Angle C changes. If C is a right angle (left figure above), then we obtain the Pythagorean Theorem as a special case:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos\left(\frac{\pi}{2}\right)}_{=0}$$

$$c^2 = a^2 + b^2$$

If C is acute (middle figure above), then:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos C}_{>0}$$

$$c^2 < a^2 + b^2$$

This reflects the fact that c in this case is smaller than c in the right angle case. If C is obtuse (right figure above), then:

$$c^2 = a^2 + b^2 - \underbrace{2ab \cos C}_{<0}$$

$$c^2 > a^2 + b^2$$

This reflects the fact that c in this case is larger than c in the right angle case.

Part C: Variations of the law

The form given in Part A is the only one you need to memorize, but you should be aware of variations. There is nothing “special” about side c and Angle C . “Role-switching” yields analogous formulas for the other side-angle pairs.

Variations of the Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Without loss of generality, the proof of one variation yields the others, as well. The previous formulas may also be solved for the cos expressions.

More Variations of the Law of Cosines

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

In radians, A , B , and C must be in the interval $(0, \pi)$, which is a subset of the range of the $\cos^{-1}$ function, $[0, \pi]$. Therefore, we can conclude that:

$$C = \cos^{-1} \left(\frac{a^2 + b^2 - c^2}{2ab} \right)$$

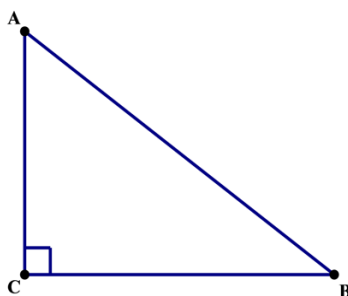
and so forth.

Part D: What must be true of all triangles?

We assume that A, B, and C are angles whose degree measures are strictly between 0° and 180° .

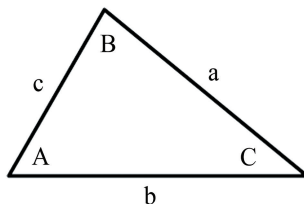
Which brings us to the main question for today: Why is it that the interior angles of a triangle always add up to 180° ? As it turns out, you can figure this out by thinking about the interior and exterior angles of a triangle. To see what mean, either grab your imagination or a sheet of paper because it's time for a little mathematical arts-and-crafts drawing project.

Start by drawing a right triangle with one horizontal leg, one vertical leg, and with the hypotenuse extending from the top left to the bottom right. Now make a copy of this triangle, rotate it around 180° , and nestle it up hypotenuse-to-hypotenuse with the original (just as we did when figuring out how to find the area of a triangle). Finally, make yet another copy of the original triangle and shift it to the right so that it's sitting right next to the newly-formed rectangle. With me so far? If so, your picture should look like this:



What's the point of this picture? Take a look at the interior angle at the bottom right of the original triangle (the one labeled "A"). Now take a look at the two angles that make up the exterior angle for that corner of the triangle (the ones labeled "B" and "C"). As we know, if we add up the interior and exterior angles of one corner of a triangle, we always get 180° . And our little drawing shows that the exterior angle in question is equal to the sum of the other two angles in the triangle. In other words, the other two angles in the triangle (the ones that add up to form the exterior angle) must combine with the angle in the bottom right corner to make a 180° angle.

For the sake of simplicity, we've made our drawing using a right triangle. But it turns out that you can make an exactly analogous drawing using any triangle you fancy, and you'll always end up reaching the same conclusion. Try making a few drawings starting with different triangles of your choosing to see this for yourself.



The inevitable conclusion of this game is that the interior angles of a triangle must always add up to 180° . Our lovely and elegant little drawing proves that this must be so.

The Cosine Rule

The cosine rule is a formula commonly used in trigonometry to determine certain aspects of a non-right triangle when other key parts of that triangle are known or can otherwise be determined. It is an effective extension of the Pythagorean theorem, which typically only works with right triangles and states that the square of the hypotenuse of the triangle is equal to the squares of the other two sides when added together ($c^2 = a^2 + b^2$). The cosine rule is an extension of this mathematic principal that makes it effective for non-right triangles and states that in regard to a certain angle, the square of the side of the triangle opposite that angle is equal to the squares of the other two sides added together, minus two times both of those sides multiplied together with the cosine of that angle ($c^2 = a^2 + b^2 - 2ab \cos C$ where C is the angle opposite side c).

Though many modern mathematical sources give credit to a Muslim mathematician named al-Kashi for the creation of the cosine rule, there is also some evidence to indicate that the ancient Greek mathematician Euclid had devised a similar principal. Much of modern algebra and trigonometry comes from the efforts of Muslims during the European Dark Ages, and it was around the 15th century that al-Kashi codified the formula in a way that is still understood today. In France the rule is even referred to as *Le théorème d'Al-Kashi* or “the theorem of al-Kashi.”

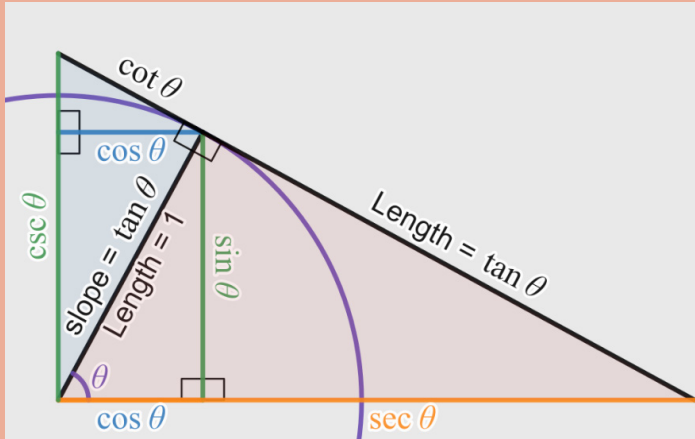
In general, the cosine rule is used in triangulation and a number of other practical applications of trigonometry. It is especially useful in systems where the lengths all three sides are known or can be established and the measure of the angles within the triangle need to be determined. The cosine rule can also be used to establish the length of one side of a triangle if the lengths of the other two sides are known as well as the angle opposite that side. Since the cosine rule deals with triangles consisting of three straight sides and their angles, it generally only works within the realm of Euclidean geometry. Different versions of the cosine rule can be used for non-Euclidean geometry such as spherical geometry and hyperbolic geometry. In these systems, a triangle is established by three points in curved space and the lines, usually curved lines, which connect them. The hyperbolic law of cosines and the spherical law of cosines function much like the Euclidean cosine rule, in that they can allow someone to establish the three angles of a triangle as long as he or she knows the three sides. Unlike Euclidean cosine rules, however, these non-Euclidean laws can also allow someone to determine the sizes of the three sides of a triangle if he or she knows the three angles.

CONCEPT OF PYTHAGOREAN IDENTITIES

The basic relationship between the sine and cosine is given by the Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1,$$

Where $\sin^2 \theta$ means $(\sin \theta)^2$ and $\cos^2 \theta$ means $(\cos \theta)^2$.



This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\begin{aligned}\sin \theta &= \pm \sqrt{1 - \cos^2 \theta}, \\ \cos \theta &= \pm \sqrt{1 - \sin^2 \theta}.\end{aligned}$$

where the sign depends on the quadrant of θ .

Dividing this identity by $\sin^2 \theta, \cos^2 \theta$ or both yields the following identities:

$$\begin{aligned}1 + \cot^2 \theta &= \csc^2 \theta \\ 1 + \tan^2 \theta &= \sec^2 \theta \\ \sec^2 \theta + \csc^2 \theta &= \sec^2 \theta \csc^2 \theta\end{aligned}$$

Using these identities, it is possible to express any trigonometric function in terms of any other.

SUMMARY

- Applications of trigonometry are applied in areas such as architecture, celestial mechanics, surveying, etc.
- Astronomy is one of the most popular fields that use trigonometry as one of the essential applications for the basic functioning of the field.
- Trigonometry is used in construction sites in measuring ground surfaces and fields. When constructing a building, aspects like roof inclination, roof slopes, perpendicular and parallel walls, light angles, sun shading, etc., require trigonometry.
- Trigonometry is used to in measuring the height of a building or a mountain. The distance of a building from the viewpoint and the elevation angle can easily determine the height of a building using the trigonometric functions.
- Trigonometry allows pilots and flight engineers to account for their speed, velocity, angle of ascent or descent, as well as other external factors such as wind speed and direction.
- The functions of trigonometry are helpful to calculate a trajectory of a projectile and to estimate the causes of a collision in a car accident.
- Trigonometry is often used by marine biologists for measurements to figure out the depth of sunlight that affects algae to photosynthesis.
- - Navigation is the process to accurately determine one's position and planning a route for the movement of a craft or vehicle from one place to another. Navigation also refers to the specialized knowledge used by navigators to perform navigational tasks.
- Trigonometry in the form of triangulation is at the heart of navigation, whether it is by land, sea, or air. GPS systems use triangulation to find and fix positions, extrapolating what is unknown from what is known.
- Surveyors use a special type of telescope called a theodolite. It is mounted on a tripod, and measures distances and angles in three-dimensional space.
- Angle of depression is a term used to describe the angle formed by two lines, one extending from an observer's eye to the horizon and another to an object located some horizontal distance away from and below the observer.

MULTIPLE CHOICE QUESTIONS

1. If the length of the shadow of a tree is decreasing then the angle of elevation is:
 - a. Increasing
 - b. Decreasing
 - c. Remains the same
 - d. None of the above
2. The angle of elevation of the top of a building from a point on the ground, which is 30 m away from the foot of the building, is 30° . The height of the building is:
 - a. 10 m
 - b. $30/\sqrt{3}$ m
 - c. $\sqrt{3}/10$ m
 - d. 30 m
3. If the height of the building and distance from the building foot's to a point is increased by 20%, then the angle of elevation on the top of the building:
 - a. Increases
 - b. Decreases
 - c. Do not change
 - d. None of the above
4. If a tower 6m high casts a shadow of $2\sqrt{3}$ m long on the ground, then the sun's elevation is:
 - a. 60°
 - b. 45°
 - c. 30°
 - d. 90°
5. The angle of elevation of the top of a building 30 m high from the foot of another building in the same plane is 60° , and also the angle of elevation of the top of the second tower from the foot of the first tower is 30° , then the distance between the two buildings is:
 - a. $15\sqrt{3}$ m
 - b. $12\sqrt{3}$ m
 - c. 36 m
 - d. $10\sqrt{3}$ m

Answer

1. (a) 2. (b) 3. (c) 4. (a) 5. (d)

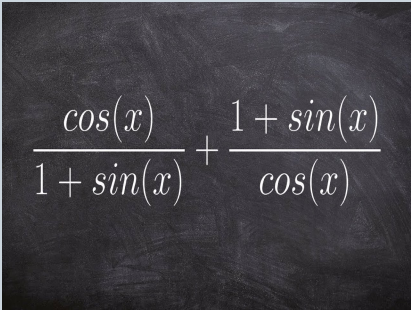
REVIEW QUESTIONS

1. How to measure height of a building or a mountain in trigonometry?
2. Define the term trigonometry in aviation.
3. What is the math review of trigonometry and navigation?
4. Explain the ambiguous case of the law of sines.
5. What do you mean by angle of depression?
6. What must be true of all triangles?

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THE FRACTIONAL SPIRAL FUNCTIONS


$$\frac{\cos(x)}{1 + \sin(x)} + \frac{1 + \sin(x)}{\cos(x)}$$

Fractional trigonometries based on the multi-valued fractional generalization of the exponential function, the R-function. The trigonometries contain the traditional trigonometric functions as proper subsets. Also developed are relationships between the R-function and the new fractional trigonometric functions. Laplace transforms are derived for the new functions and are used to generate solution sets for various classes of fractional differential equations.

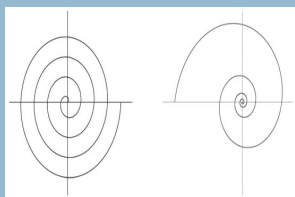
After studying this chapter, you will understand the core concepts of:

- The fractional spiral functions
- Analysis of spirals

INTRODUCTION

Fractional spiral functions are a class of mathematical functions that generalize the concept of a spiral. They are expressed using fractional exponents or parameters, which allows for more flexibility in defining the curvature and shape of the spiral. These functions are often used in mathematics, physics, and various scientific disciplines to model complex patterns and behaviors.

Many spiral functions occur in mathematics: the logarithmic or logistic spiral, the hyperbolic, parabolic, and Cornu spirals, and so on. The mathematical relationships between these spirals and the fractional spirals described here have not yet been studied. Because of their frequent occurrence in nature, there have been several books that have studied spirals and their occurrences.



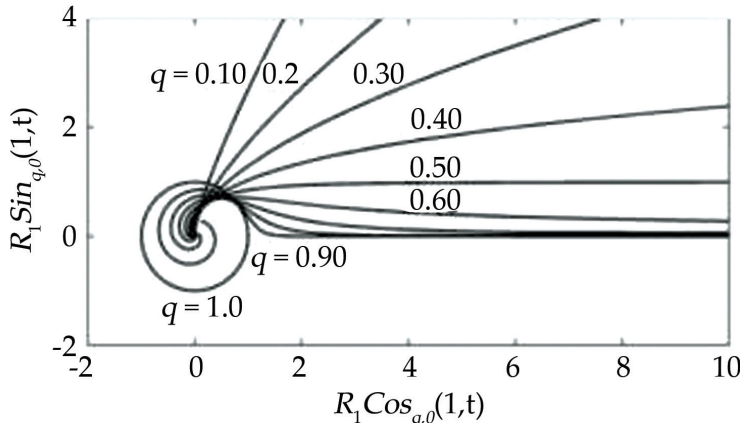
CORE CONCEPT: THE FRACTIONAL SPIRAL FUNCTIONS

In the ordinary trigonometry, it is well known that when $x = \cos(t)$ is plotted against $y = \sin(t)$, the result is a circle of radius one in the x - y plane. Here, we study some of the phase-plane behavior of the fractional trigonometric functions.

In Figure 1, $R_1 \sin_{q,v}(1, 0, t)$ is plotted as a function $R_1 \cos_{q,v}(1, 0, t)$ for various values of the order, q , between zero and one. Note that when $q = 1$, $v = 0$, we obtain the well-known circular functions. However, when $q \neq 1$, we obtain a more general set of plots, the R_1 -Spiral Functions.

Parametrically reducing q in Figure 1 from $q = 1$ to $q = 0.1$ generalizes the circular functions to the set of fractional spirals that traverse from infinity to the origin. The fractional trigonometry releases us from the radius constraint of the **unit circle**, which is the basis of the classical trigonometry! The R_1 -spiral functions contain the circular functions as a special case.

Figure 2 is the same plot but with $1 \leq q \leq 2$. Here, the spirals move outward from the origin. Note that the growth rate increases as q increases. The morphology of the R_1 -spiral functions in the range of $1 \leq q \leq 2$ is not seen in the classical spirals and may be unique. This morphology is denoted as a “barred spiral” in astronomy.



Unit circle is a circle on the Cartesian plane that has a radius of 1 unit and is centered at the origin (0, 0).

Figure 1: Phase plane plot, $R_1 \cos_{q,v}(1, 0, t)$ versus $R_1 \sin_{q,v}(1, 0, t)$ for $q=0.1-1$ in steps of 0.1. With $a=1$, $v=0$, $t=0-7.4$.

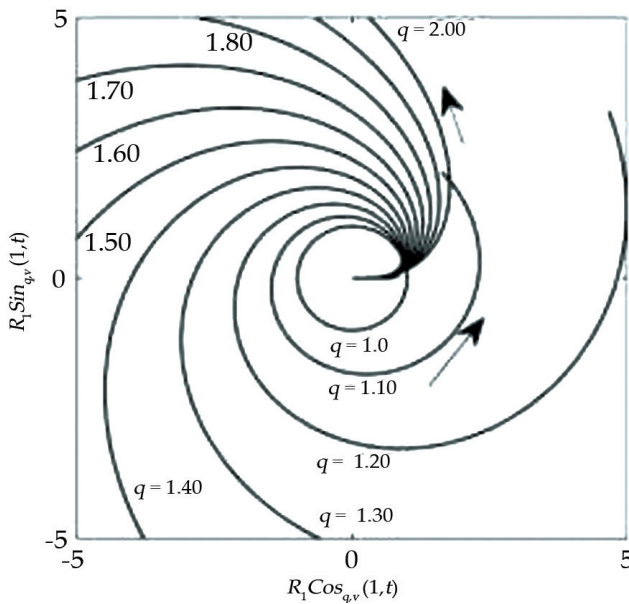


Figure 2: Phase plane plot, $R_1 \cos_{q,v}(1, t)$ versus $R_1 \sin_{q,v}(1, t)$. Effect of q , with $q=1.0-2.0$ in steps of 0.1, $a=1.0$, $v=0$, $t=0-7.4$. Arrows indicate t increasing.

The R_2 -trigonometric functions also have a very special spiral-like behavior in the phase plane. With $v=0$, when the order variable q is between zero and two, that is, $0 < q < 2$, the phase plane loci are always attracted to circles. In Figure 3a, for $0.2 \leq q < 1$, the loci arrive from infinity and are attracted to circles of radius $1/q$. In Figure 3b, when $1 < q < 2$, the loci emanate from the origin and are also attracted to circles of radius $1/q$. Figure 3c shows the loci spiraling outward toward infinity when $q > 2$.

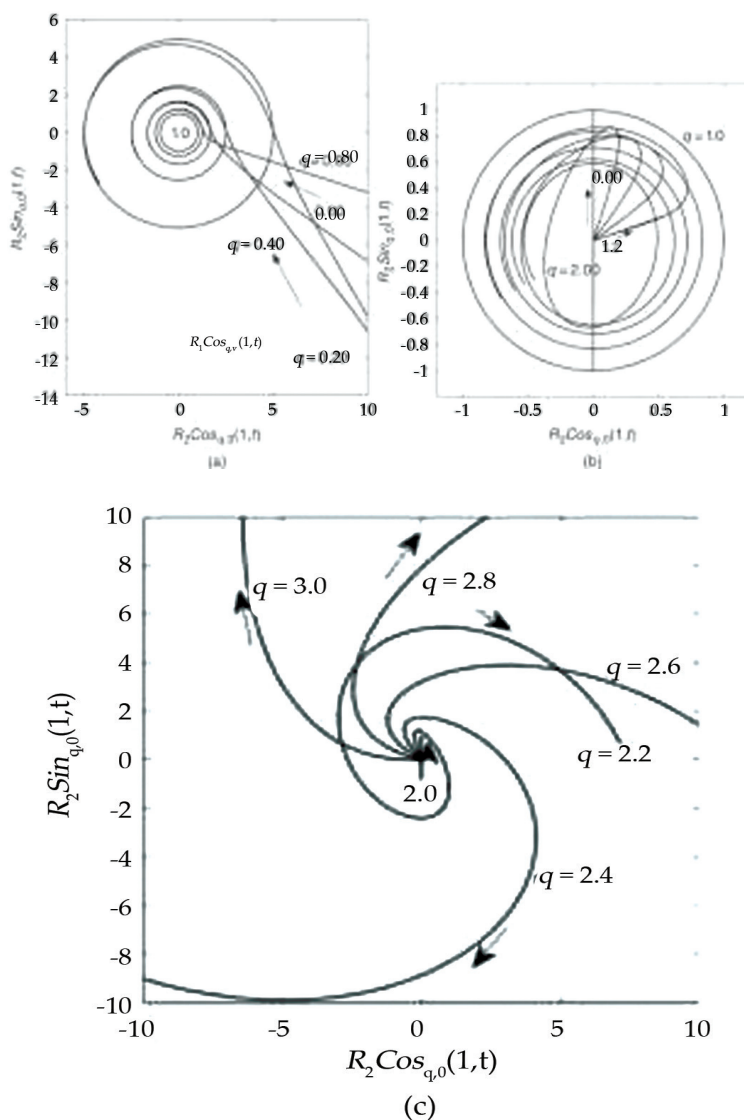


Figure 3: Phase plane plots, $R_2 \cos_{q,v}(1, 0, t)$ versus $R_2 \sin_{q,v}(1, 0, t)$ for $q=0.2$ – 3 in steps of 0.2 with $a=1$, $v=0$.

Phase plane plots for $R_3 \cos_{q,v}(1, 0, t)$ and $R_3 \sin_{q,v}(1, 0, t)$ for values of q between 1.0 and 5.0 are shown in Figures 4 and 5. The spiral growth rate is seen to rapidly increase with increases in q . The effect of v on selected R_1 -functions is shown in Figure 6.

The three-dimensional loci $R_2 \cos_{38/39,0}(1, k, h)$ versus $R_2 \sin_{38/39,0}(1, k, h)$ versus h are shown in Figure 7. Four views along the major axes of the figure show the effect of variation of the index variable, k . Because the order variable q has been selected to be $q=38/39$, which is near unity, we have an example of the “near-order effect”. Strong symmetry may be observed when the Cos plane is viewed from the Sin axis and in the Cos versus Sin phase plane. This figure is strongly suggestive of tornadic or cyclonic weather forms.

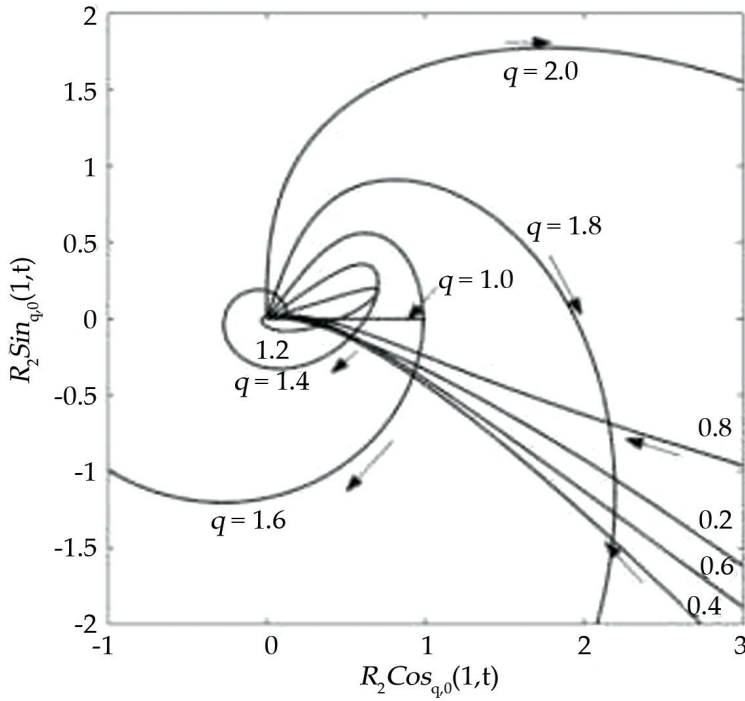


Figure 4: Phase plane plot for $R_3\text{Cos}_{q,0}(1, t)$ versus $R_3\text{Sin}_{q,0}(1, t)$ for $q=0.2-2$ in steps of 0.2 , $v=0$, $a=1$, $k=0$, $t=0-10$.

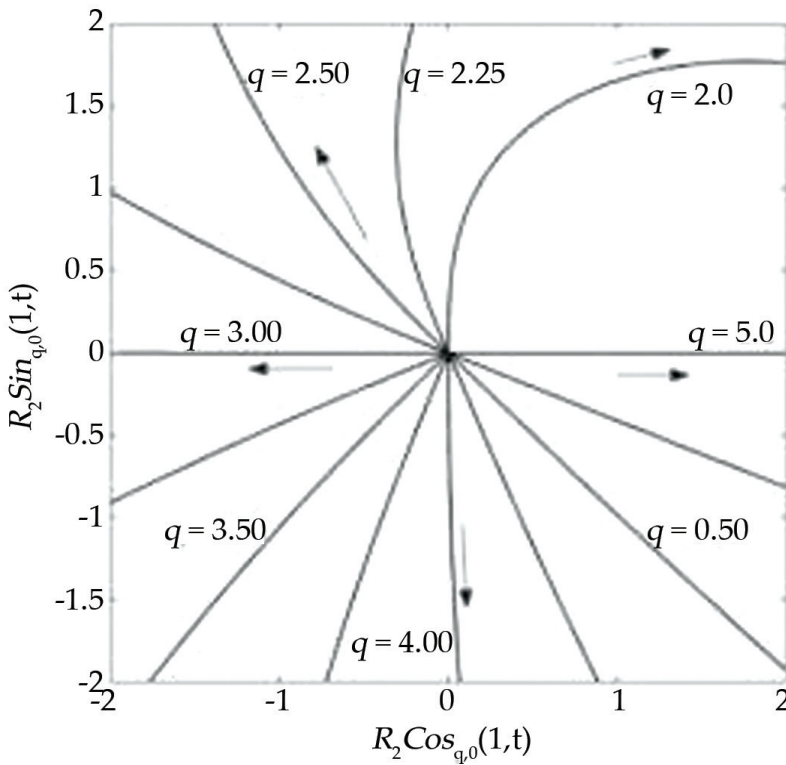


Figure 5: Phase plane plot for $R_3\text{Cos}_{q,0}(1, t)$ versus $R_3\text{Sin}_{q,0}(1, t)$ for $q=2-5$ in steps of 0.25 , $v=0$, $a=1$, $k=0$, $t=0-4$.

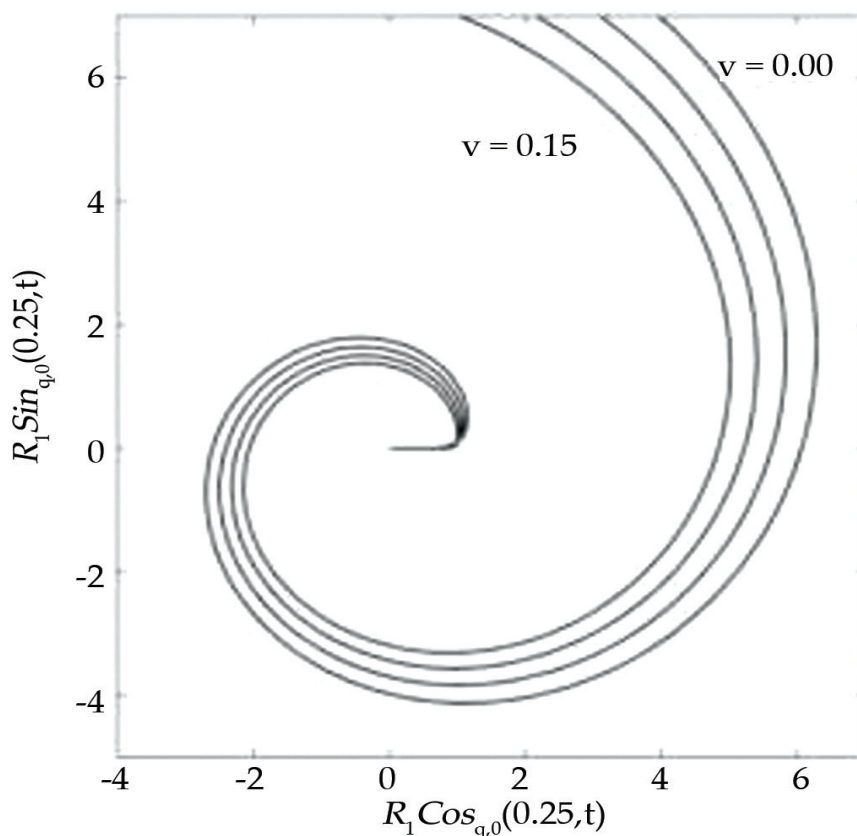


Figure 6: Effect of v on $R_1 \text{Cos}_{1.20,v}(0.25, t)$ versus $R_1 \text{Sin}_{1.20,v}(0.25, t)$, phase plane plot with $q=1.20$, $v=0-0.15$ in steps of 0.05 , $a=0.25$, $k=0$, $t=0-28$.

Figure 8 is another four part view. Panels a–d show four views of $R_2 \text{Coflut}_{3839,0}(1, k, h)$ versus $R_2 \text{Covib}_{3839,0}(1, k, h)$ versus h , for $k=0-38$. The unique aspect of this figure is the apparent symmetry seen from all three axes. Symmetry seen in Figure 8b–d and when viewed along the diagonals (view not shown) is stunning.

The fractional meta-trigonometric functions are rich with spiral behavior.

The foundation for fractional trigonometry was laid by mathematicians like Leonhard Euler and Joseph Fourier in the 18th century, who introduced the concept of fractional derivatives and integrals. Euler, in particular, worked on developing the theory of differentiation and integration with non-integer orders.

A small selection of these functions is shown in Figures 9–13. Figure 9 shows the effect of α on the phase plane plot for $\text{Sin}_{0.8,-1}(1, \alpha, 0.5, t)$ versus $\text{Cos}_{0.8,-1}(1, \alpha, 0.5, t)$. The loci for this set are unusual for the fractional trigonometric functions in that they do not terminate at infinity, the origin, or in a circular orbit. Figures

10 and 11 are phase plane plots for the complexity trigonometric functions $\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ and $-\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $-\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ (dashed). These are typical fractional spirals found in various domains of the fractional

trigonometry. In Figure 10, for $q < 1$, the loci spiral outward. Figures 12 and 13 show some spiral functions found for the parity fractional trigonometric functions. Observe how the β parameter controls the exit angle as the loci leave the end of the bars. The loci in Figure 14 are not spirals; however, the figure is of interest because the loci leave the origin with “double-barred” behavior, a very unusual phenomenon.

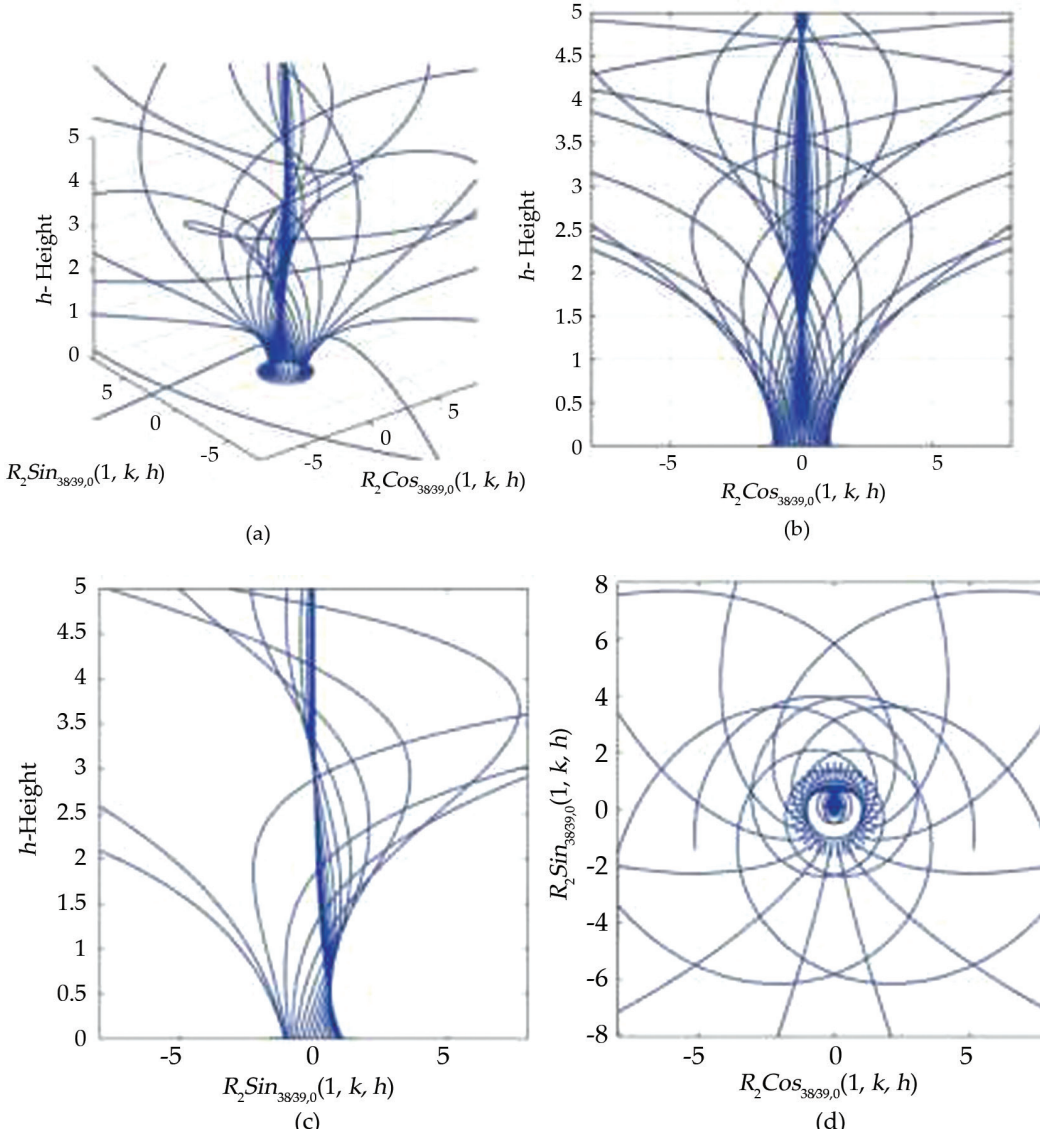
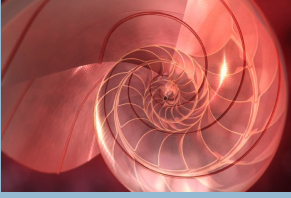


Figure 7: Four views of $R_2 \cos_{38/39,0}(1, k, t)$ versus $R_2 \sin_{38/39,0}(1, k, t)$ versus h for $k = 0$ – 38 .



CORE CONCEPT: ANALYSIS OF SPIRALS

A general consideration of spiral behavior is pursued. The objective is the development of a general analysis capability for spirals. This will allow comparison of the fractional spirals with some of the classical spirals and enable further exploration of the relationships of the fractional spiral functions to the circular functions.

Descriptions of Spirals

There are several ways in which **spirals**, and other plane curves, may be described. We shall consider the polar and parametric descriptions.

Spiral is a curve which emanates from a point, moving farther away as it revolves around the point. It is a subtype of whorled patterns, a broad group that also includes concentric objects.

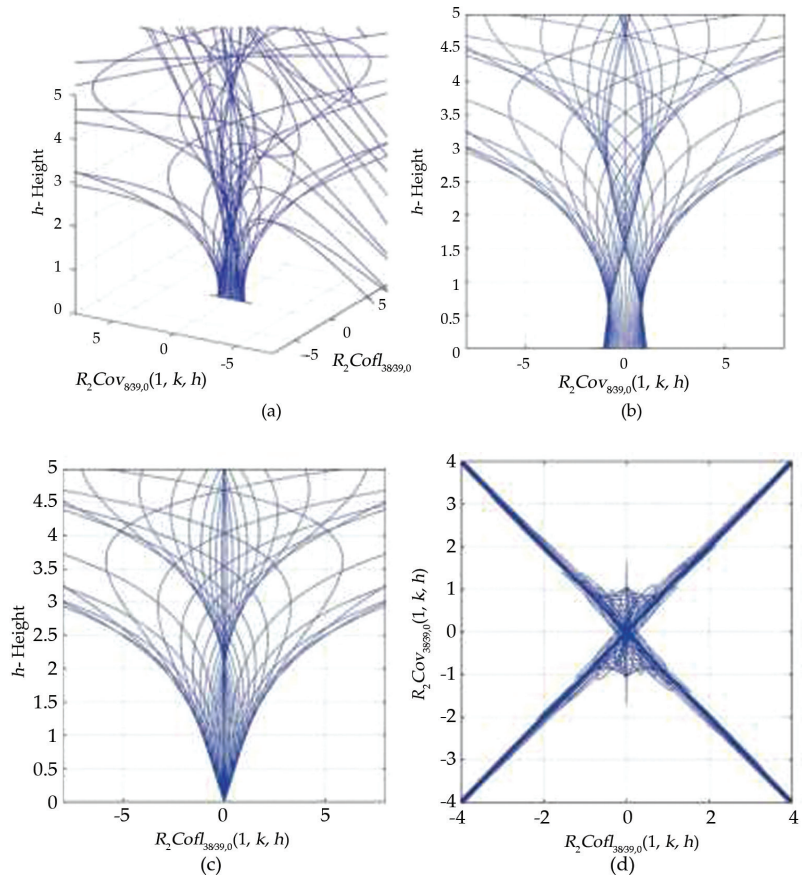


Figure 8. (a–d). Four views of $R_2Coflut_{3839,0}(1, k, h)$ versus $R_2Covib_{3839,0}(1, k, h)$ versus h , for $k=0-38$. With $a=1.0$, and $v=0$.

Polar Description

The polar representation of a spiral, or more generally of a **plane curve**, is given by

$$r = f(\theta) \text{ or equivalently, } \theta = g(r), \quad (1)$$

where r is a radial distance from the origin, or other point of reference, and θ is an angle of rotation that is positive in the counterclockwise direction; see Figure 15.

Plane curve is a curve in a plane that may be either a Euclidean plane, an affine plane or a projective plane.

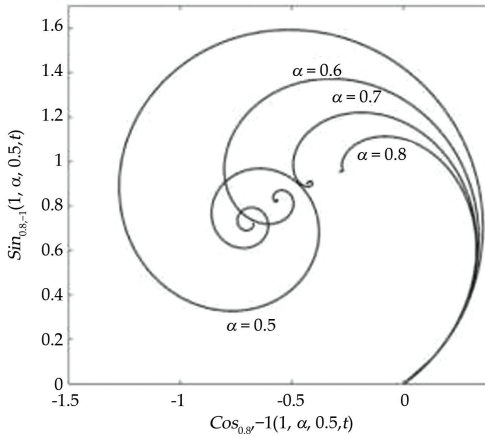


Figure 9. Phase plane plot for $\text{Sin}_{0.8,-1}(1, \alpha, 0.5, t)$ versus $\text{Cos}_{0.8,-1}(1, \alpha, 0.5, t)$ for $q=0.8$, $v=-1$, $a=1.0$, $\alpha = 0.5-0.8$ in steps of 0.1 , $\beta = 0.5$, $k = 0$ and $t = 0-20$.

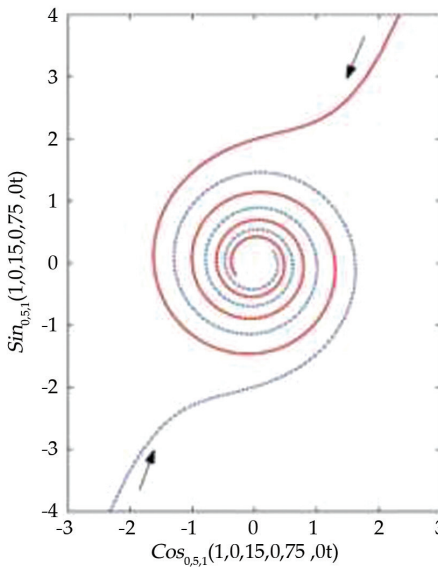


Figure 10: Phase plane plot for $\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ and $-\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $-\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ (dashed) for $q=0.5$, $v=1$, $a=1.0$, $\alpha = 0.15$, $\beta = 0.75$, $k=0$ and $t = 0-22$.

Parametric Description

The parametric description of a spiral, or a plane curve, expresses the rectangular (or polar) coordinates of the curve in terms of a parametric value, for example, t , thus

$$x = x(t) \text{ and } y = y(t), \text{ or } r = r(t) \text{ and } \theta = \theta(t). \quad (2)$$

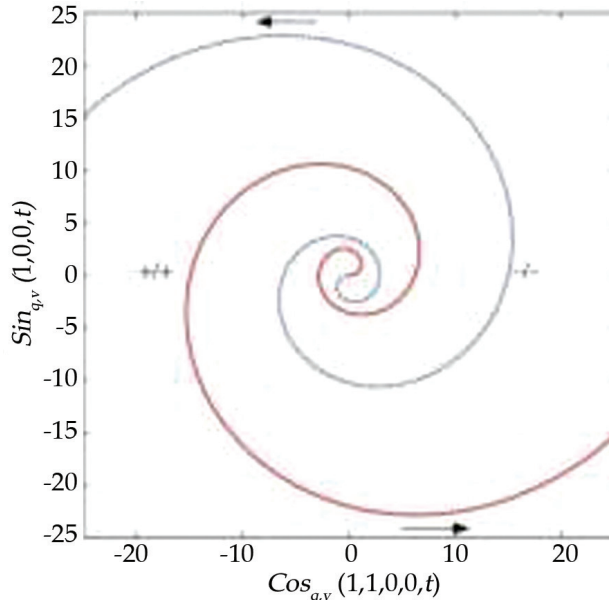


Figure 11: Phase plane plot for $\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ and $-\text{Cos}_{q,v}(a, \alpha, \beta, k, t)$ versus $-\text{Sin}_{q,v}(a, \alpha, \beta, k, t)$ (dashed) for $q=1.2$, $v=-0.8$, $a=1.0$, $\alpha=1.0$, $\beta=0$, $k=0$ and $t=0-14$.

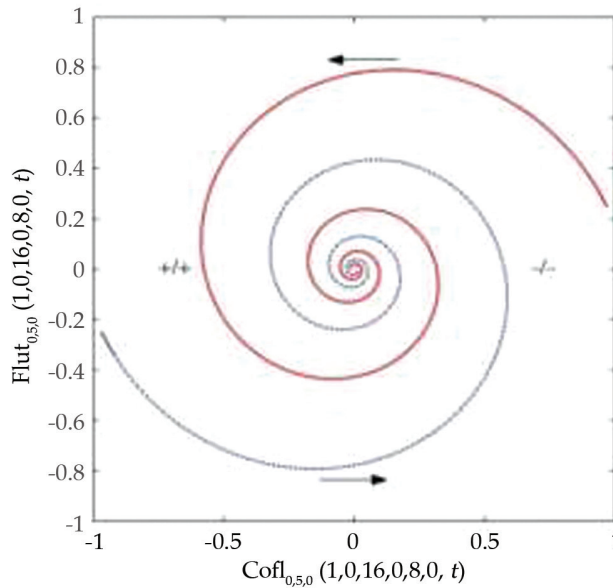


Figure 12: Phase plane plot for $\text{Flut}_{q,v}(a, \alpha, \beta, k, t)$ versus $\text{Cofl}_{q,v}(a, \alpha, \beta, k, t)$ and $-\text{Flut}_{q,v}(a, \alpha, \beta, k, t)$ versus $-\text{Cofl}_{q,v}(a, \alpha, \beta, k, t)$ (dashed) for $q=0.5$, $v=0$, $a=1.0$, $\alpha=0.16$, $\beta=0.80$, $k=0$ and $t=0-22$.

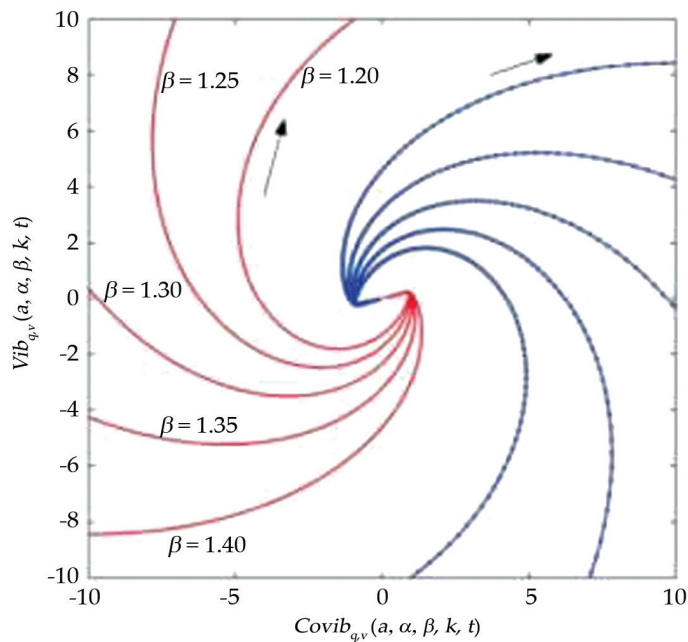


Figure 13: Phase plane plot for $Vib_{q,v}(a, \alpha, \beta, k, t)$ versus $Covib_{q,v}(a, \alpha, \beta, k, t)$ and $-Vib_{q,v}(a, \alpha, \beta, k, t)$ versus $-Covib_{q,v}(a, \alpha, \beta, k, t)$ (dotted) for $q=1.07$, $v=-0.070$, $a=1.0$, $\alpha=0.02$, $\beta=1.2-1.4$ in steps of 0.05, $k=0$ and $t=0-10$.

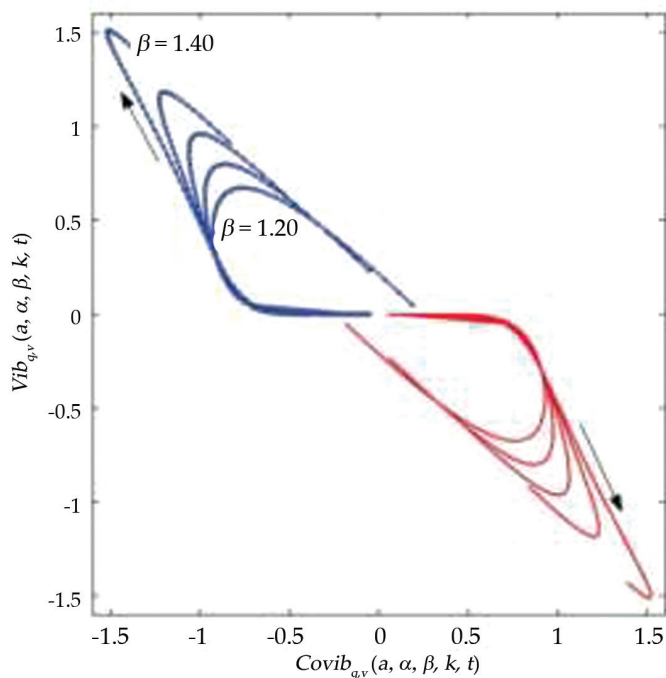


Figure 14: Phase plane plot for $Cofl_{q,v}(a, \alpha, \beta, k, t)$ versus $Covib_{q,v}(a, \alpha, \beta, k, t)$ and $-Cofl_{q,v}(a, \alpha, \beta, k, t)$ versus $-Covib_{q,v}(a, \alpha, \beta, k, t)$ (dotted) for $q=1.06$, $v=-0.01$, $a=1.0$, $\alpha=0.05$, $\beta=1.2-1.4$ in steps of 0.05, $k=0$ and $t=0-2.5$.

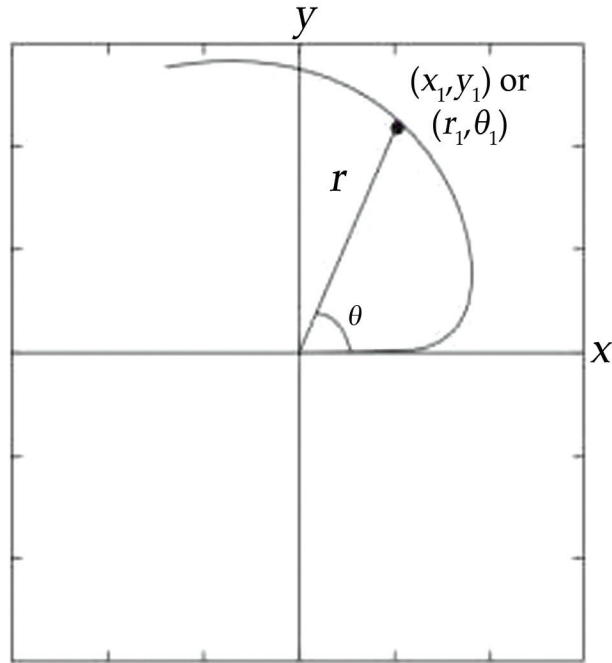


Figure 15: Coordinate systems for both polar and parametric forms.

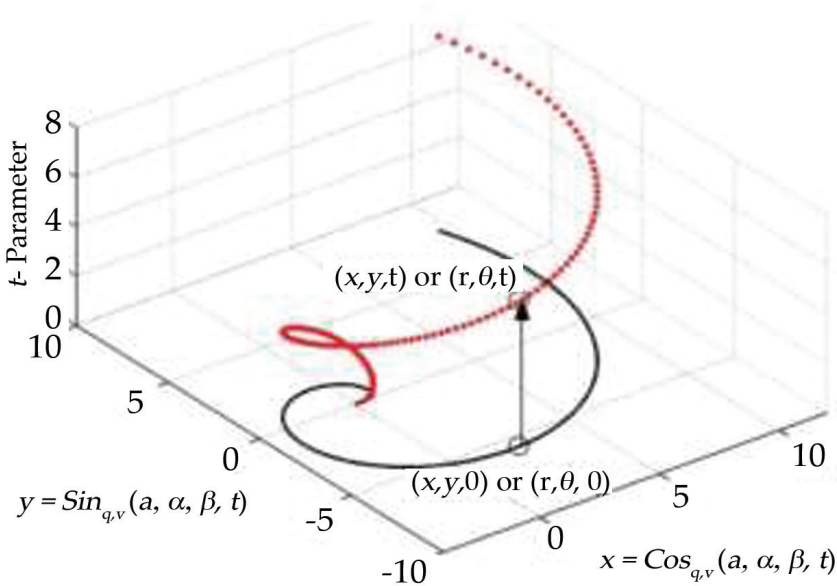


Figure 16: Parametric spiral interpreted as a three-dimensional space curve.

The conversion between the polar and rectangular forms is described by the well-known relationships:

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1} \left(\frac{y}{x} \right) \quad \text{or} \quad (3)$$

$$x = r \cos(\theta) \quad \text{and} \quad y = r \sin(\theta). \quad (4)$$

The parametric description may also be considered as a three-dimensional space

curve in three dimensions x , y , and t . This is illustrated in Figure 16 for the spiral defined by the equations

$$x = R_1 \cos_{1.3,0}(1, 0, t) \text{ and } y = R_1 \sin_{1.3,0}(1, 0, t), \quad (5)$$

with $t = 0-7.9$.

Definitions

Davis describes a search for a meaningful and mathematically useful definition for spirals. Specialized definitions for a particular purpose have been used; however, few generalized definitions are found. One such definition is found in his notes p. 205. It is attributed to G. Fouret, " $r = f(\theta)(\theta + g(\theta))$ where f and g are complex trigonometric functions of θ ."

Alternate Definitions

For simplicity, here we consider only the origin as the basis of rotation, and only consider plane spirals.

Let R be a radius vector in a polar coordinate system and θ the polar angle measured from the x -axis. Then, a simple ccw (counter clockwise) spiral is defined by the function $r(\theta)$, where $r(\theta)$ is a continuous non-negative **monotonic function** with $\theta \geq 0$ and $\theta \rightarrow \infty$. A general, ccw, spiral is defined as a function $R(\theta)$, where

$$r_1(\theta) \leq R(\theta) \leq r_2(\theta), \quad \forall \theta \geq 0, \quad (6)$$

where $r_1(\theta)$ and $r_2(\theta)$ are simple ccw spirals, typically with no common points other than the origin (or other point of reference). Note that $R(\theta)$ may be a discontinuous and/or **multivalued function** with only the constraint that it be bounded by the simple spirals. Typically, we could also constrain $R(\theta)$ to be a continuous function. This definition allows considerable latitude in the types of functions considered to be spiral while maintaining the general spirit of what a spiral is. For spirals that wrap in a clockwise (cw) manner, the definition is as follows: A simple cw (clockwise) spiral is defined by the function $r(\theta)$, where $r(\theta)$ is a continuous nonpositive monotonic function with $\theta \leq 0$ and $\theta \rightarrow -\infty$.

Monotonic function is a function between ordered sets that preserves or reverses the given order.

Multivalued function is a set-valued function with continuity properties that allow considering it locally as an ordinary function.

A general, cw, spiral then is defined as a function $R(\theta)$, where

$$r_1(\theta) \leq R(\theta) \leq r_2(\theta), \quad \forall \theta \leq 0, \quad (7)$$

where $r_1(\theta)$ and $r_2(\theta)$ are simple, cw, spirals, typically, with no common points other than possibly the origin (or other point of reference). Furthermore, $R(\theta)$ may be a discontinuous and/or multivalued function with only the constraint that it is bounded by the simple spirals. From these definitions, a mathematical theory of spirals, that is, addition, multiplication, and so on, is possible.

Multivalued functions of a complex variable have branch points. For example, for the n th root and logarithm functions, 0 is a branch point; for the arc-tangent function, the imaginary units i and $-i$ are branch points. Using the branch points, these functions may be redefined to be single-valued functions, by restricting the range. A suitable interval may be found through use of a branch cut, a kind of curve that connects pairs of branch points, thus reducing the multilayered Riemann surface of the function to a single layer. As in the case with real functions, the restricted range may be called the principal branch of the function.

Examples

Consider the three curves defined by $r_1 = 0.5 \exp(0.3\theta)$, $r_2 = 1.0 \exp(0.3\theta)$, and $r_3 = 0.2 \exp(0.3\theta) \sin(5\theta) + \sin(5\theta) + \frac{(r_1+r_2)}{2}$.

This spiral is shown in Figure 17a. For the spiral in Figure 17a, the r_1 and r_2 curves are simple ccw spirals, and the r_3 curve is seen to be a continuous general ccw spiral (over the range shown). Figure 17b shows a general ccw spiral composed of a random sequence of points bounded by the same r_1 and r_2 simple ccw spirals.

Spiral Length and Growth/Decay Rates

Spiral Length

The length of an arc is well known from the classical calculus. It is given in rectangular coordinates by the equation

$$L = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy, \quad (8)$$

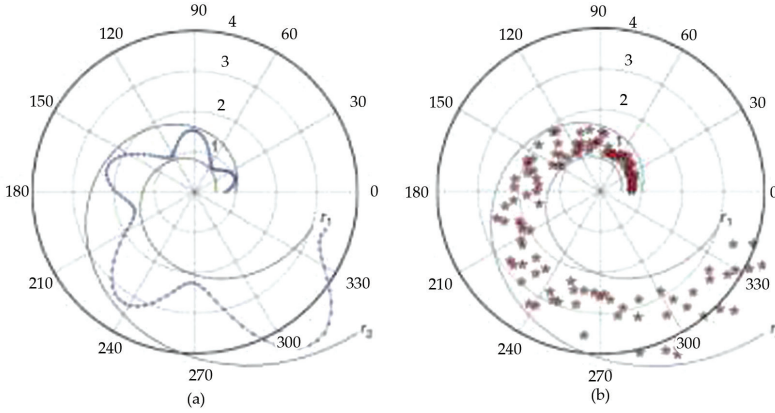


Figure 17: Polar plot showing two simple counterclockwise spirals bounding (a) a general continuous and (b) a general discontinuous counterclockwise spiral.

where it is assumed that $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is a continuous bounded function. In polar coordinates, the arc length is given as

$$L = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

where again it is assumed that $\frac{dr}{d\theta}$ is a continuous bounded function. The parametric form will be most useful for application to the fractional spirals, that is,

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt, \quad (9)$$

where again the derivatives are continuous bounded functions. For application to an R_1 -spiral, for example, the relationships $\frac{d}{dt}R_1 \cos_{q,v}(a, t) = R_1 \cos_{q,v+1}(a, t)$ and $\frac{d}{dt}R_1 \sin_{q,v}(a, t) = R_1 \sin_{q,v+1}(a, t)$ are applied to an R_1 -spiral to provide the length from the origin to t_{final} as

$$L(t) = \int_{t_{\text{ini}}=0+}^{t_{\text{fin}}} \left((R_1 \cos_{q,v+1}(a, t))^2 + (R_1 \sin_{q,v+1}(a, t))^2 \right)^{1/2} dt, \quad (10)$$

Spiral Growth Rates for ccw Spirals

The rate of growth or decay of spirals described using the polar representation (equation (1)) is given by

$$\text{Radial Growth Rate} = \frac{dr}{d\theta}, \quad (11)$$

where a positive value indicates spiral growth and a negative value indicates decay for ccw spirals. When the spiral is described by the parametric form, the growth rate can also be set in those terms. Thus, based on equation (3),

$$dr(t) = d(x^2(t) + y^2(t))^{1/2} = \frac{x(t)dx(t) + y(t)dy(t)}{(x^2(t) + y^2(t))^{1/2}}, \quad (12)$$

and the time rate of growth of the radius is

$$\frac{dr}{dt} = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{(x^2(t) + y^2(t))}}. \quad (13)$$

For the spiral angle

$$d\theta(t) = d\left(\tan^{-1}\left(\frac{y(t)}{x(t)}\right)\right) = \frac{x(t)dy(t) - y(t)dx(t)}{x^2(t) + y^2(t)}. \quad (14)$$

Similarly, the time rate of growth of the angle is given as

$$\frac{d\theta(t)}{dt} = \frac{x(t)y'(t) - y(t)x'(t)}{x^2(t) + y^2(t)}. \quad (15)$$

Combining the results of equations (12) and (14) gives

$$\frac{dr(t)}{d\theta(t)} = \frac{(x(t)dx(t) + y(t)dy(t))(x^2(t) + y^2(t))^{1/2}}{x(t)dy(t) - y(t)dx(t)}. \quad (16)$$

Similarly, from the time rates, equations (13) and (15), we have

$$\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \frac{(x(t)x'(t) + y(t)y'(t))(x^2(t) + y^2(t))^{1/2}}{x(t)y'(t) - y(t)x'(t)}. \quad (17)$$

Component Growth Rates

It is necessary in some analysis to use the component growth rates; these are obtained as follows. Differentiating $x = r \cos(\theta)$ and $y = r \sin(\theta)$ gives

$$dx = \cos(\theta)dr - r \sin(\theta)d\theta, \quad dy = \sin(\theta)dr + r \cos(\theta)d\theta. \quad (18)$$

Now, $r = f(\theta) \Rightarrow dr = f'(\theta)d\theta$ in the aforementioned equations

$$\begin{aligned} \frac{dx}{d\theta} &= \cos(\theta)f'(\theta) - r \sin(\theta), \quad \frac{dy}{d\theta} = \sin(\theta)f'(\theta) + r \cos(\theta) \text{ or} \\ \frac{dx}{d\theta} &= \cos(\theta)f'(\theta) - f(\theta) \sin(\theta), \quad \frac{dy}{d\theta} = \sin(\theta)f'(\theta) + f(\theta) \cos(\theta). \end{aligned} \quad (19)$$

Alternatively, $\theta = g(r) \Rightarrow d\theta = g'(r)dr$ in equation (18) gives

$$\begin{aligned} \frac{dx}{dr} &= \cos(\theta) - rg'(r) \sin(\theta), \quad \frac{dy}{dr} = \sin(\theta) + rg'(r) \cos(\theta) \text{ or} \\ \frac{dx}{dr} &= \cos(g(r)) - rg'(r) \sin(g(r)), \quad \frac{dy}{dr} = \sin(g(r)) + rg'(r) \cos(g(r)). \end{aligned} \quad (20)$$

Scaling of Spirals

We consider the process of scaling spiral functions; consideration shall be done in both polar and rectangular forms. The parameters of the original (reference) spiral are subscripted as $x_o(t)$ and $y_o(t)$, while the scaled versions are given as $x_s(t)$ and $y_s(t)$, and so on.

Uniform Rectangular Scaling

In uniform rectangular scaling, the parametric or rectangular values defining the spiral are multiplied by the same scale factor, k . Thus,

$$x_s = kx_o \quad \text{and} \quad y_s = ky_o. \quad (21)$$

The polar variables are

$$r_s = \sqrt{(kx_o)^2 + (ky_o)^2} = k\sqrt{x_o^2 + y_o^2} \quad \text{or} \\ r_s = kr_o \quad (22)$$

or

$$\tan \theta_s = \frac{ky_o}{kx_o} = \tan \theta_o \Rightarrow \theta_s = \theta_o. \quad (23)$$

Since $dr_s = kdr_o$ and $d\theta_s = d\theta_o$, we have

$$\frac{dr_s}{d\theta_s} = k \frac{dr_o}{d\theta_o}. \quad (24)$$

Thus, we see that the growth rate of the uniformly scaled spiral increases or decreases proportional to the value of the scale factor, k .

Nonuniform Rectangular Scaling

In nonuniform rectangular scaling, different factors are used to multiply the parametric or rectangular values of the defining the spiral. Thus,

$$x_s = k_x x_o \quad \text{and} \quad y_s = k_y y_o, \quad (25)$$

and the polar variables are

$$r_s = \sqrt{(k_x x_o)^2 + (k_y y_o)^2} = k_x \sqrt{x_o^2 + \left(\frac{k_y y_o}{k_x}\right)^2} = k_y \sqrt{\left(\frac{k_x x_o}{k_y}\right)^2 + y_o^2}, \\ r_s = k_x x_o \sqrt{1 + \left(\frac{k_y y_o}{k_x x_o}\right)^2} = k_x x_o \sqrt{1 + \left(\frac{k_y}{k_x}\right)^2 \tan^2(\theta_o)}, \\ r_s = k_y y_o \sqrt{1 + \left(\frac{k_x x_o}{k_y y_o}\right)^2} = k_y y_o \sqrt{1 + \left(\frac{k_x}{k_y}\right)^2 \cot^2(\theta_o)}. \quad (26)$$

Solving for $k_x x_o$ and $k_y y_o$ from these equations, we may write

$$x_o = \frac{r_s}{k_x \sqrt{1 + \left(\frac{k_y}{k_x}\right)^2 \tan^2(\theta_o)}}, \quad (27)$$

$$y_o = \frac{r_s}{k_y \sqrt{1 + \left(\frac{k_x}{k_y}\right)^2 \cot^2(\theta_o)}}, \quad (28)$$

$$r_o = \sqrt{(x_o)^2 + (y_o)^2} = \sqrt{\frac{r_s^2}{k_x^2 + k_y^2 \tan^2(\theta_o)} + \frac{r_s^2}{k_y^2 + k_x^2 \cot^2(\theta_o)}}, \quad (29)$$

$$r_s = \frac{r_o}{\sqrt{\frac{1}{k_x^2 + k_y^2 \tan^2(\theta_o)} + \frac{1}{k_y^2 + k_x^2 \cot^2(\theta_o)}}},$$

$$\tan(\theta_s) = \frac{k_y y_o}{k_x x_o} = \frac{k_y}{k_x} \tan(\theta_o), \quad (30)$$

$$\theta_s = \tan^{-1} \left(\frac{k_y}{k_x} \tan(\theta_o) \right). \quad (31)$$

Therefore, the nonuniform scaling distorts the x, y relationship, it scales the radius and rotates the spiral. It is noted that if $k_x = k_y = k$ in equations (28) and (29) the equations reduce to the uniform scaling equations (22) and (23).

Polar Scaling

Here, we apply scaling factors to the polar variables and examine the effect on the rectangular variables.

Radial Scaling

In radial scaling, we scale the radius while keeping the polar angle fixed, that is,

$$r_s = k_r r_o \text{ and } \theta_s = \theta_o, \text{ then}$$

$$r_s = k_r \sqrt{x_o^2 + y_o^2} = \sqrt{(k_r x_o)^2 + (k_r y_o)^2} = \sqrt{x_s^2 + y_s^2}. \quad (32)$$

One possible relationship pair to satisfy equation (32) is

$$x_s = k_r x_o, \quad y_s = k_r y_o, \quad (33)$$

that is, the condition resulting from uniform rectangular scaling. Thus, we see radial scaling is equivalent to uniform rectangular scaling.

Angular Scaling

In angular scaling, the radius is fixed while the angle is scaled; thus,

$$\theta_s = k_a \theta_o, \quad r_s = r_o. \quad (34)$$

Now,

$$x_s = r_s \cos(\theta_s), \quad y_s = r_s \sin(\theta_s),$$

and

$$x_o = r_o \cos(\theta_o), \quad y_o = r_o \sin(\theta_o).$$

Since $r_s = r_o$,

$$x_s = \left(\frac{\cos(k_a \theta_o)}{\cos(\theta_o)} \right) x_o, \quad y_s = \left(\frac{\sin(k_a \theta_o)}{\sin(\theta_o)} \right) y_o. \quad (35)$$

Thus, we see that the scaling of the x and y coordinate varies with θ_s .

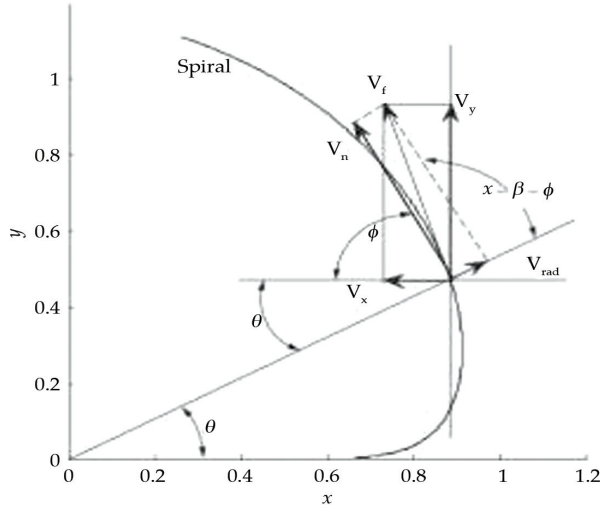


Figure 18: Velocity components.

Spiral Velocities

If a spiral, such as that shown in Figure 18, is considered as a trajectory, then velocities along the trajectory may be determined when the t parameter is treated as time. Our goal here is treatment of the fractional spiral functions; however, we start with a general treatment for spirals expressed rectangular parametric form. Based on the parametric definition (equation (2)), the x and y components of velocity are defined as:

$$V_x(t) = \lim_{\delta t \rightarrow 0} \left\{ \frac{x(t + \delta t) - x(t)}{\delta t} \right\} = \frac{dx(t)}{dt}, \quad (36)$$

$$V_y(t) = \lim_{\delta t \rightarrow 0} \left\{ \frac{y(t + \delta t) - y(t)}{\delta t} \right\} = \frac{dy(t)}{dt}. \quad (37)$$

Thus, the resultant velocity is given by

$$V_r(t) = \sqrt{V_x^2(t) + V_y^2(t)}, \quad (38)$$

and the associated direction is expressed as

$$\varphi(t) = \tan^{-1} \left(\frac{V_y(t)}{V_x(t)} \right). \quad (39)$$

Now, the slope of the spiral at time equal t is $M = dy(t)/dx(t)$. The angle associated with this slope is given by

$$\beta = \tan^{-1} \left(\frac{dy(t)}{dx(t)} \right) = \tan^{-1} \left(\frac{y'(t)dt}{x'(t)dt} \right) = \varphi(t),$$

and we see that the resultant velocity is tangent to the spiral at time t .

It is also of interest to determine the radial and normal components of the resultant velocity relative to the vector from the origin to the spiral at time t . If θ is the angle from the positive x axis to the point $x(t)$, $y(t)$; see Figure

18. Then, the radial and normal components, $V_{rad}(t)$ and V_n , respectively, are determined as follows:

$$V_{rad}(t) = V_r(t) \cos(\pi - \theta - \phi) = -V_r(t) \cos(\theta + \phi), \quad (40)$$

where ϕ is defined by equation (39). Applying the sum of angles formula for the cosine function from the ordinary trigonometry,

$$\begin{aligned} V_{rad}(t) &= -V_r(t) \{ \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi) \}, \\ V_{rad}(t) &= -V_r(t) \left\{ \cos \left(\tan^{-1} \left(\frac{y(t)}{x(t)} \right) \right) \cos \left(\tan^{-1} \left(\frac{V_y}{V_x} \right) \right) - \right. \\ &\quad \left. \sin \left(\tan^{-1} \left(\frac{y(t)}{x(t)} \right) \right) \sin \left(\tan^{-1} \left(\frac{V_y}{V_x} \right) \right) \right\}, \\ V_{rad}(t) &= -V_r(t) \left\{ \frac{x(t)}{\sqrt{x^2 + y^2}} \frac{V_x}{\sqrt{V_x^2 + V_y^2}} - \frac{y(t)}{\sqrt{x^2 + y^2}} \frac{V_y}{\sqrt{V_x^2 + V_y^2}} \right\}, \\ V_{rad}(t) &= -V_r(t) \left\{ \frac{V_x(t)x(t) - V_y(t)y(t)}{\sqrt{V_x^2(t) + V_y^2(t)} \sqrt{x^2(t) + y^2(t)}} \right\}. \end{aligned} \quad (41)$$

But $\sqrt{x^2 + y^2} = r$, the radial distance from the origin to the point $x(t)$, $y(t)$, and $\sqrt{V_x^2 + V_y^2} = V_r$, giving

$$V_{rad}(t) = \frac{V_y(t)y(t) - V_x(t)x(t)}{r}. \quad (42)$$

The normal velocity is determined in similar manner, that is,

$$V_n(t) = V_r(t) \sin(\pi - \theta - \phi) = V_r(t) \sin(\theta + \phi). \quad (43)$$

Applying the sum of angles formula for the sine from the ordinary trigonometry

$$\begin{aligned} V_n(t) &= V_r(t) \left\{ \sin \left(\tan^{-1} \left(\frac{y(t)}{x(t)} \right) \right) \cos \left(\tan^{-1} \left(\frac{V_y(t)}{V_x(t)} \right) \right) \right. \\ &\quad \left. + \cos \left(\tan^{-1} \left(\frac{y(t)}{x(t)} \right) \right) \sin \left(\tan^{-1} \left(\frac{V_y(t)}{V_x(t)} \right) \right) \right\}, \end{aligned} \quad (44)$$

$$V_n(t) = V_r(t) \left\{ \frac{y(t)}{\sqrt{x^2 + y^2}} \frac{V_x}{\sqrt{V_x^2 + V_y^2}} + \frac{x(t)}{\sqrt{x^2 + y^2}} \frac{V_y}{\sqrt{V_x^2 + V_y^2}} \right\}, \quad (45)$$

When applied to say, the R1-spiral functions, we have

$$x(t) = R_1 \cos_{q,v}(a, t),$$

$$y(t) = R_1 \sin_{q,v}(a, t),$$

$$\begin{aligned} V_x(t) &= \lim_{\delta t \rightarrow 0} \left\{ \frac{R_1 \cos_{q,v}(a, t + \delta t) - R_1 \cos_{q,v}(a, t)}{\delta t} \right\} \\ &= {}_0D_t^1 R_1 \cos_{q,v}(a, t) = R_1 \cos_{q,v+1}(a, t), \end{aligned} \quad (46)$$

$$\begin{aligned} V_y(t) &= \lim_{\delta t \rightarrow 0} \left\{ \frac{R_1 \sin_{q,v}(a, t + \delta t) - R_1 \sin_{q,v}(a, t)}{\delta t} \right\} \\ &= {}_0D_t^1 R_1 \sin_{q,v}(a, t) = R_1 \sin_{q,v+1}(a, t), \end{aligned} \quad (47)$$

where the differentiation rules (equations (36) and (40)) have been applied. The velocities for the R_1 -spiral functions then may be obtained by substitution of these values into equations (33), (42), and (44); thus,

$$V_r(t) = \sqrt{R_1^2 \cos_{q,v+1}^2(a, t) + R_1^2 \sin_{q,v+1}^2(a, t)}, \quad (48)$$

$$V_{rad}(t) = \frac{R_1 \sin_{q,v+1}(a, t) R_1 \sin_{q,v}(a, t) - R_1 \cos_{q,v+1}(a, t) R_1 \cos_{q,v}(a, t)}{\sqrt{R_1^2 \cos_{q,v}^2(a, t) + R_1^2 \sin_{q,v}^2(a, t)}}, \quad (49)$$

$$V_n(t) = \frac{R_1 \cos_{q,v+1}(a, t) R_1 \sin_{q,v}(a, t) + R_1 \sin_{q,v+1}(a, t) R_1 \cos_{q,v}(a, t)}{\sqrt{R_1^2 \cos_{q,v}^2(a, t) + R_1^2 \sin_{q,v}^2(a, t)}}. \quad (50)$$

These results may be readily extended to the meta-trigonometric spirals.

Referenced Spirals: Retardation

It is useful at times to compare spirals either from the same family or of different families. It is of particular interest to compare with a special spiral, the zero growth spiral, that is, the unit circle. The unit circle is, of course, the basis of the circular functions and is the geometric locus for the $q=1$, $v=0$, R_1 -trigonometry. As a spiral, it is obtained when $x(t) = \cos(t)$ and $y(t) = \sin(t)$, where typically t is considered to be the time variable. We observe that $t = \tan^{-1}(y(t)/x(t))$, for $0 \leq t \leq 2\pi$, that is, for the first revolution. Thus, for this situation, t represents both t -time the parameter and t the geometric **angle**. This is not the typical case for the R_1 - and R_2 -spiral functions, or the meta-trigonometric spirals. Note, the results and definitions apply only to positive evolving or counterclockwise developing spirals. Obvious changes are required for clockwise evolving spirals.

In Figure 19, we compare spirals that have evolved to the same time t . The angles, α , associated with these spirals are in general different. When, for the same value of the parametric variable t , the angle of a test spiral is smaller than that for the reference spiral to which it is being compared, we consider it to be “retarded,” and if larger it is considered to be “advanced.” We define

$$\text{Angle of Retardation } \lambda_r(t) \equiv \alpha_{ref}(t) - \alpha_{test}(t). \quad (51)$$

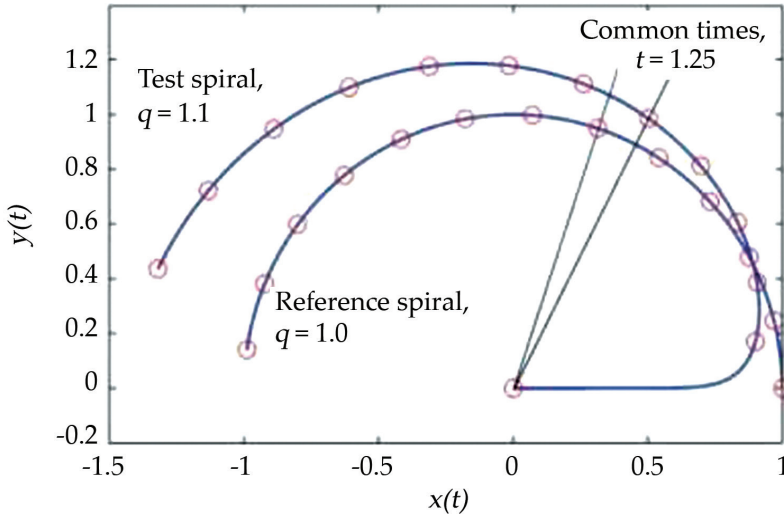


Figure 19: Angular retardation demonstrated for spiral defined by $R_1 \text{Trig}_{q,v}(a, t)$ with $q=1.1$ for test spiral and $q=1.0$ for reference spiral; for both spirals, $a=1$, $k=0$, and $v=0$, $x(t) = R_1 \text{Cos}_{q,0}(a, t)$ and $y(t) = R_1 \text{Sin}_{q,0}(a, t)$.

If the unit circle is considered to be the reference spiral in this instance, then the retardation angle λ_r is given by

$$\lambda_r(t) = t - \alpha_{test}(t). \quad (52)$$

Spirals may also be compared based on the value of the parameter t (time) for spirals that have evolved to a common angle. Thus, with reference to Figure 20, the retardation time is defined as

$$\text{Retardation Time} = \delta t_r(\alpha) \equiv t_{ref}(\alpha) - t_{test}(\alpha). \quad (53)$$

Again if the unit circle is considered to be the reference spiral, then the retardation time δt_r is given by

$$\delta t_r(\alpha) = t - t_{test}(\alpha). \quad (54)$$

When the retardation time is positive, $\delta t_r(\alpha) < 0$, the test spiral is “late or retarded” relative to the reference spiral, and conversely it is “early or advanced” when $\delta t_r(\alpha) > 0$.

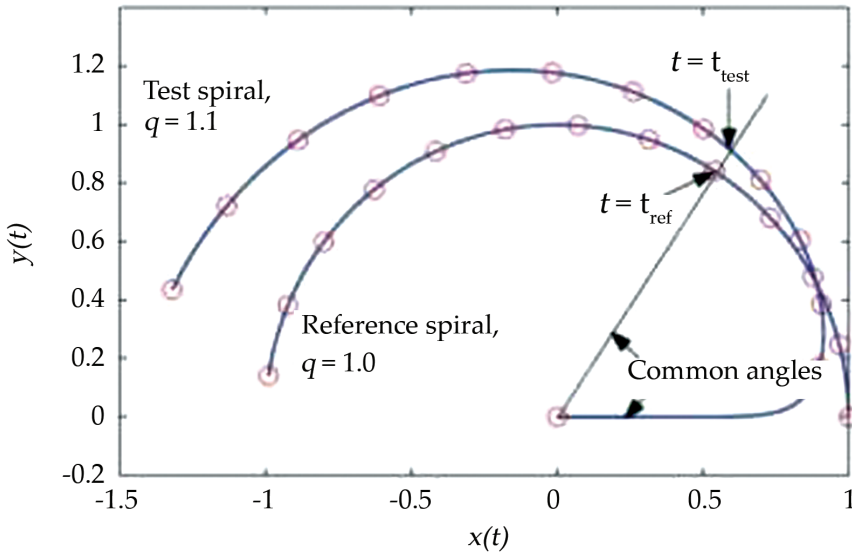


Figure 20: Time retardation demonstrated for spiral defined by $R_1 \text{Trig}_{q,v}(a, t)$ with $q=1.1$ for test spiral and $q=1.0$ for reference spiral; for both spirals, $a=1$, $k=0$, and $v=0$, $x(t) = R_1 \text{Cos}_{q,0}(a, t)$ and $y(t) = R_1 \text{Sin}_{q,0}(a, t)$.

We now consider, as an example, the R_1 family of spirals. The retardation angle is given by

$$\lambda_r = \alpha_{ref} - \alpha_{test} = \tan^{-1}(R_1 \text{Tan}_{qref,vref}(a_{ref}, t)) - \tan^{-1}(R_1 \text{Tan}_{qtest,vtest}(a_{test}, t)),$$

where the reference spiral is taken to be a general member of the R_1 family. If the reference spiral is taken to be the unit circle, this becomes

$$\lambda_r = \alpha_{ref} - \alpha_{test} = t - \tan^{-1}(R_1 \text{Tan}_{qtest,vtest}(a_{test}, t)) \quad (55)$$

For the retardation time of the R_1 family of spirals, we need a common reference angle α on which to base the comparison. For the test and reference spirals, these are given by

$$\tan(\alpha) = R_1 \text{Tan}_{qtest,vtest}(a_{test}, t_{test}) = R_1 \text{Tan}_{qref,vref}(a_{ref}, t_{ref}),$$

and the related times are

$$t_{test} = R_1 \text{Tan}_{qtest,vtest}^{-1}(a_{test}) \text{ and } t_{ref} = R_1 \text{Tan}_{qref,vref}^{-1}(a_{ref}). \quad (56)$$

Note that explicit forms for the $R_1 \text{Tan}_{q,v}^{-1}$ are not available; thus, numerical computations are required. The retardation time $\delta t_r(\alpha)$ is then determined a

$$\delta t_r(\alpha) \equiv t_{ref}(\alpha) - t_{test}(\alpha) = R_1 \text{Tan}_{qref,vref}^{-1}(a_{ref}) - R_1 \text{Tan}_{qtest,vtest}^{-1}(a_{test}). \quad (57)$$

If the reference spiral is taken to be the unit circle, the retardation time becomes

$$\delta t_r(\alpha) \equiv t_{ref}(\alpha) - t_{test}(\alpha) = t - R_1 \text{Tan}_{qtest,vtest}^{-1}(a_{test}) \quad (58)$$

CONCEPT OF MITTAG-LEFFLER FUNCTION

In mathematics, the Mittag-Leffler function $E_{\alpha,\beta}$ is a special function, a complex function which depends on two complex parameters α and β . It may be defined by the following series when the real part of

α is strictly positive:

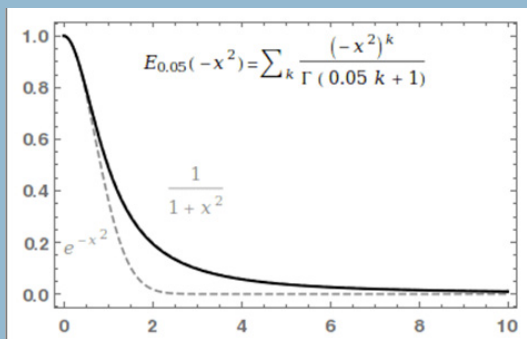
$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\Gamma(x)$ is the gamma function. When $\beta = 1$, it is abbreviated as

$E_{\alpha}(z) = E_{\alpha,1}(z)$. For $\alpha = 0$ the series above equals the Taylor expansion of the geometric series and consequently

$$E_{0,\beta}(z) = \frac{1}{\Gamma(\beta)} \frac{1}{1-z}$$

In the case α and β are real and positive, the series converges for all values of the argument z , so the Mittag-Leffler function is an entire function. This function is named after Gösta Mittag-Leffler. This class of functions are important in the theory of the fractional calculus.



For $\alpha > 0$, the Mittag-Leffler function $E_{\alpha,1}(z)$ is an entire function of order $1/\alpha$, and is in some sense the simplest entire function of its order.

The Mittag-Leffler function satisfies the recurrence property

$$E_{\alpha,\beta}(z) = \frac{1}{z} E_{\alpha,\beta-\alpha}(z) - \frac{1}{z\Gamma(\beta-\alpha)},$$

from which the Poincaré asymptotic expansion

$$E_{\alpha,\beta}(z) \sim -\sum_{k=1}^{\infty} \frac{1}{z^k \Gamma(\beta - k\alpha)}$$

follows, which is true for $z \rightarrow -\infty$.

SUMMARY

- Fractional spiral functions are a class of mathematical functions that generalize the concept of a spiral.
- The fractional meta-trigonometric functions are rich with spiral behavior.
- In uniform rectangular scaling, the parametric or rectangular values defining the spiral are multiplied by the same scale factor, k .
- The fractional trigonometry releases us from the radius constraint of the unit circle, which is the basis of the classical trigonometry! The R_1 -spiral functions contain the circular functions as a special case.
- In nonuniform rectangular scaling, different factors are used to multiply the parametric or rectangular values of the defining the spiral.
- It is useful at times to compare spirals either from the same family or of different families. It is of particular interest to compare with a special spiral, the zero growth spiral, that is, the unit circle.

MULTIPLE CHOICE QUESTIONS

1. If $\sin t = 1/5$ and $0 < t < \pi/2$, then $\cos(4t) = ?$
 - a. 0.3464
 - b. 0.8
 - c. 0.6928
 - d. -0.6928
2. If $\cos t = 1/3$ and $3\pi/2 < t < 2\pi$ in quadrant IV, then $\sin(4t) = ?$
 - a. $8\sqrt{2}/3$
 - b. $-8\sqrt{2}/3$
 - c. $-56\sqrt{2}/243$
 - d. $56\sqrt{2}/81$
3. Which of the following is not correct?
 - a. $\sin(x) = -\sin(-x)$
 - b. $\sin(\pi + x) = \sin(x)$
 - c. $\sec(-t) = \sec(t)$
 - d. $\cos(\pi - x) = -\cos(x)$
4. Which of the following is not the same as $\tan(t)$?
 - a. $-\tan(-t)$
 - b. $\tan(t + 2\pi)$
 - c. $\tan(t + \pi)$
 - d. $\tan(t + \pi/2)$
5. If point (a, b) is on the Unit Circle associated with the rotation t and point (c, d) is also on the Unit circle associated with rotation $t + \pi$, then which of the following is correct?
 - a. $c = -a$ and $d = -b$
 - b. $c = -a$ and $d = b$
 - c. $c = a$ and $d = b$
 - d. $c = a$ and $d = -b$

Answer

1. (c) 2. (d) 3. (b) 4. (d) 5. (a)

REVIEW QUESTIONS

1. What do you understand by the fractional spiral functions? Explain.
2. Discuss about coordinate systems for both polar and parametric forms.
3. What is the mathematical description of a spiral?
4. Measure spiral growth rates for ccw spirals.
5. Differentiate between uniform rectangular scaling and nonuniform rectangular scaling.

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Trigonometry deals with the relationships between angles and sides of triangles, plays a crucial role in various fields. From engineering to physics and even music, its applications are extensive. In the realm of engineering, trigonometry is indispensable. Engineers rely on this subject to design structures such as bridges and buildings. By using trigonometric functions like sine, cosine, and tangent, they can accurately calculate angles and distances. This enables them to ensure stability, efficiency, and safety in their designs. Moreover, trigonometry finds widespread use in physics. It helps physicists analyze forces acting on objects by breaking them down into components along different axes. The knowledge of trigonometric principles assists them in understanding complex motions involving rotation or oscillation accurately. Furthermore tangents can be found within sound waves while studying acoustics—another application area for Trigonometry importance lies here too inside audio production becoming significant due to frequency response calculations ensuring clean reception composition. Another interesting field where trigonometry shines is astronomy! Astronomers utilize this mathematical tool when calculating celestial positions such as determining stellar parallax or measuring the distance between galaxies—and it's an essential aspect more generally throughout space research.

Core Concepts in Trigonometry (2nd Edition) presents students and self-learners with a lucid and comprehensible exploration of trigonometry's fundamental principles. It delves into the examination of angles and dimensions within triangles. Although this may appear straightforward, trigonometry plays an integral role in modern engineering, design, architecture, and various other disciplines. Moreover, it consistently furnishes students and educators with well-structured explanations of mathematical concepts. The application of trigonometry in geometry involves the computation of side lengths and angles primarily within right-angled triangles. Geothermal applications serve as the most prevalent utilization of trigonometry whereby information about a triangle can unveil precise measurements pertaining to its sides and angles. This book aims to cultivate your ability to think mathematically efficiently. Through its guidance, you can transcend mere rote memorization or imitation of examples—instead fostering genuine problem-solving skills that will endure throughout time.

Duried Alwazeer, an experienced Assistant Professor in the Department of Mathematics, excels in teaching a range of topics including but not limited to: the Pythagorean Theorem, special right triangles, and basic trigonometric functions, solving right triangles, measuring rotation and applying trig functions to angles of rotation. With his profound knowledge and expertise, he guides students towards understanding the principles behind Trigonometric Functions of Any Angle as well as relating them effectively. Duried has actively participated and presented at numerous conferences and seminars throughout his career. Additionally, he is a prolific author with an impressive collection consisting of 20 research papers along with 5 book chapters under his name. Furthermore, he has published two notable books that have contributed significantly to his academic credit.